The Benefit of Collective Reputation*

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Abstract

We study a model of collective reputation and use it to analyze the benefit of collective brands. Consumers form beliefs about the quality of an experience good that is produced by one firm that is part of a collective brand. Consumers' limited ability to distinguish among firms in the collective and to monitor firms' investment decisions creates incentives to free-ride on other firms' investment efforts. Nevertheless, we show that collective brands induce stronger incentives to invest in quality than individual brands if the main concern is with the acquisition of specialized knowledge and the baseline reputation of the collective is high or if the main concern is with quality cont rol and the baseline reputation is low. We also contrast the socially optimal information structure with the profit maximizing choice of branding if branding is endogenous. Our results can be applied to country-of-origin, agricultural appellation, and other collective brands.

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1 Introduction

A product's country of origin indicates something about its quality. How does such a collective brand or reputation operate, and how does it sustain its brand value? Why, for example, does the car manufacturer Volkswagen advertise "The power of German engineering" and so many successful Chinese suppliers emphasize their country of origin, while the German appliance manufacturer Bosch uses the non-country specific slogan "made for life"?

A brand, defined as "a unique design, sign, symbol, words, or combination of these, employed in creating an image that identifies a product and differentiates it from its competitors,"[1] can be thought of as a means to build a good reputation. When building reputation, a firm faces a moral hazard problem; its investment in quality is unobservable to current consumers, and the reputational return on its investment can only be collected in the future.

The costs and benefits of good reputation differ in a collective brand and an individual brand. At first glance, collective brands may seem like a bad idea. If several firms operate under one brand name, each firm has an incentive to free-ride on other firms' investments. Moreover, a firm's investment in its own quality has a weaker effect on the brand value of a collective brand because it may be difficult for consumers to verify the identity of individual firms within the brand, and in any case they are likely to be matched with other firms in the brand. In other words, the "precision" of the signal that is generated by a firm's investment in quality is lower in a collective brand, which weakens the incentive to invest in quality.

Nevertheless, under some circumstances, a collective reputation can serve as a commitment device for investment in high quality. If a brand is very successful (possibly as a result of previous large investments), then a firm might be discouraged from additional investment because the returns from it become small. The firm can afford to become complacent or rest on its laurels, so to speak. Analogously, if a brand develops a bad reputation (possibly as a result of no investment), then returns on investment are also low, and the firm might give up investment altogether. As we show, collective brands can mitigate these "discouragement

[1] This definition is taken from BusinessDictionary.Com.

effects" faced by individual firms after very good or very bad histories by making extreme beliefs about the value of the brand less likely.

Exactly how extreme the beliefs about a brand can be depends both on the structure of signals that consumers obtain about firms' investments in quality and the baseline reputation of firms in the industry. For example, in industries that require some exclusive knowledge about, for example, a technology, a good quality realization reveals that a firm possesses that technology and is a "competent type." In contrast, in manufacturing, quality control is important, and consumers can easily learn that a firm is incompetent (or has failed to invest) when a product has been observed to have a low quality.

We analyze a model of reputation that can nest both individual brands and collective brands with multiple firms in the vein of Mailath and Samuelson (2001). The model has the following features. Time is discrete. There are two types of firms, competent and incompetent. In every period, only competent firms have the option to invest in quality. Consumers observe the qualities of past products, which are noisy signals of past investment decisions. Given these features, competent types can differentiate themselves from incompetent types by investing over time and producing higher quality products. If consumers believe that competent types invest, then they infer that a firm with good signals is indeed more likely to be competent. As a result, they are willing to pay more for goods produced by firms with better past signals. This, in turn, provides an incentive for a competent firm to invest in quality.

Accordingly, we define a firm's reputation as the consumers' posterior belief that it is competent. The best possible equilibrium from a welfare point of view is the one in which competent firms invest after each and every history, and we call the equilibrium in which this is the case the *reputational equilibrium*. In most of the paper, we restrict our analysis to the properties of this equilibrium. As pointed out by Mailath and Samuelson (2001) for the case of an individual reputation, such an equilibrium exists only if beliefs are bounded. If beliefs are not bounded, then as the competent type continues to invest and to generate favorable

signals, consumers eventually learn almost perfectly that the firm must be competent. This destroys the firm's incentive to invest, which leads to a collapse of the reputational equilibrium. This cannot happen in our model because we assume that consumers' memory is finite and limited to the last T periods only, as in Moav and Neeman (2010).[2] This assumption captures the nature of the market's limited memory and/or the inattention paid to the very distant past. In the main part of the paper, we focus on the case where $T = 2$, but we show that our results hold for any finite $T \geq 2$.[3]

The timing of our model of collective reputation, which is a natural extension of the basic model of individual reputation that is used in the literature, is the following. Firms' types are independently drawn from a given distribution once and for all. In each period, a short-lived consumer is randomly matched with one firm. If firms establish themselves as individual brands, then consumers observe firm-specific past signals. If firms establish themselves as a collective brand, then consumers observe signals only at the brand level, and they cannot tell whether the signal is generated by the firm with which they have been matched in any specific period.

The reputational equilibrium exists in these environments if the benefit of investment exceeds its cost after every possible history. A firm has both short-run and long-run incentives to invest or refrain from investment. In the *short-run*, a firm may want to exploit its current reputation. In the *long-run*, a firm may want to build its reputation or possibly free-ride on future efforts by itself and other members of the brand. Collective reputation can improve the short-run incentives to invest because the best possible collective reputation is weaker than the best possible individual reputation, and so the incentive to cash in on existing good reputation is weaker. However, the very fact that the best possible individual reputation

[2]Mailath and Samuelson (2001) assume instead that firms exit the market and are replaced by new firms whose types are drawn from some distribution. Another alternative is to assume that a firm's type changes randomly over time, as discussed in Holmström (1999). In this paper, we assume instead that memory is bounded because it allows us to obtain closed form solutions for the parameters that admit the existence of the reputational equilibrium, which we use to compare the two models of reputation we consider: individual and collective.

[3]The case where $T = 1$ is too simple and does not allow a firm to develop an history dependent reputation.

is better than the best possible collective reputation also implies that individual reputation induces stronger incentives to invest in the long-run. It allows the firm to establish a stronger individual reputation in the future, and the firm does not have the option of free-riding on another firm's investments as in the case of collective reputation.

In the case of "exclusive knowledge," (in which a good signal reveals competence) a collective brand can help overcome the firm's moral hazard problem, in particular after a good signal. This is because a subsequent observation of a bad signal in the next period moves beliefs significantly more in the case of a collective brand: with an individual brand even after a bad signal, the consumer is still sure that the firm is competent as it has produced a high quality product in the past. In the case of a collective brand, despite the fact that one firm has been shown to be competent, other firms in the brand may be incompetent, so more good signals increase the brand value. Thus, a collective brand induces stronger incentives to invest than an individual brand if a good signal reveals competence and the baseline reputation is high. Analogously, in the case of "quality control," (in which a bad signal exposes incompetence) the discouragement effect is stronger for individual brands after a firm has produced low quality because in this case consumers infer that the firm is likely to be incompetent, regardless of what else is observed or will be observed in the future.

The fact that short-run considerations favor collective reputation and long-run considerations favor individual reputation implies that in order for collective reputation to function as a commitment device, investment decisions cannot be made too frequently, or equivalently, the discount factor cannot be too large. Thus, the short-run advantage of collective reputation can only outweigh the long-run free-riding incentive if firms do not care too much about the future.

Finally, we also address the issue of brand formation. If firms can freely choose with whom to brand, then it is important to understand whether the commitment value of a collective brand is sufficiently strong to encourage a competent firm to brand with an incompetent firm. We show that in an economy in which the prior probability that a firm is competent

is high, a competent firm in an industry that requires exclusive knowledge always wants to brand with other firms irrespective of whether they are competent or not. In contrast, in the quality control case, firms never want to brand with other firms in an economy in which this prior probability is small, even though it is socially optimal to do so. This suggests that in developing countries, government enforcement of country of origin labeling can be useful for quality control industries if moral hazard is a major concern.

The paper is structured as follows. In the next section we discuss the related literature. In Section 3, we set up the model and discuss the equilibrium concept. Section 4 presents an example that highlights the main trade-offs and intuition of our analysis. In Section 5, we investigate the existence of the reputational equilibrium for individual and collective brands separately and then show under which circumstances collective brands can serve as a commitment device. We analyze firms' incentives to brand together in Section 6 and the case where $T \geq 2$ in Section 7.1. Finally, Section 9 concludes with a discussion of the interpretation and implications of our results. All proofs are relegated to the Appendix.

2 Related Literature

Our work is related to the theoretical economics literature on reputation, as well as to the literature on umbrella branding, country-of-origin and career concerns.

The theoretical work on reputation that is most relevant to our paper is Mailath and Samuelson (2001) who formulated the insight that good individual reputation can be sustained only if consumers' beliefs about the type of the firm are bounded. As explained above, instead of assuming that firms are replaced with an exogenously given probability in every period as in Mailath and Samuelson (2001), we assume that consumers have finite memories as in Moav and Neeman (2010). This allows us to solve for the threshold cost below which firms invest in quality, which is impossible to do in Mailath and Samuelson's model.

Research that identifies the benefits of collective reputation is scarce. Tirole (1993) is

probably the first who formalized an analytical model of collective reputation in context of a large organization. In his model, a group's reputation is an aggregate of the reputations of the individual members of the group. Each group member's reputation is determined by its noisily observed past behavior as well as the group's track record. The complementarity between the group's reputation and current incentives of its members gives rise to multiple steady state equilbria. Given a good track record of the group, members have incentives to maintain the good reputation, which sustains a "low corruption" steady state equilibrium. But, starting with a bad record, the group is locked into a "high corruption" steady state equilibrium.[4]

More recently, Fishman et al. (2014) consider a two-period model in which an individual firm can only generate one signal per period. Firms' investment decision is made once-and-for-all in the first period. A collective brand that includes many firms can send many signals and so provide better information to consumers. This informational benefit outweighs firms' incentive to free-ride on other firms' investment efforts provided the number of brand members is not too large. This model abstracts away from issues of commitment and dynamic trade-offs, which are the we focus of our analysis.

Collective reputation has also been studied in the context of umbrella branding in which an existing brand name is extended to a new product line. Wernerfelt (1988), Choi (1998), Cabral (2000), Miklós-Thal (2012), and Moorthy (2012) have examined the incentives that a monopolist has to signal quality by pooling reputation for different products. Others have considered settings where free-riding incentives are more pronounced. Zhang (2015) examines country-of-origin labelling. He shows that the ability to free-ride on other firms' quality investments implies that high quality firms have an incentive to dissociate themselves from the country-of-origin label, which in turn mitigates free-riding and can improve the reputation for the group. Bar-Isaac (2007) studies an overlapping generations model in the moral hazard-in-teams (career concerns) framework developed by á la Holmström (1982).

[4]Levin (2009) extends Tirole (1993) by considering the case where the cost of high effort evolves stochastically over time.

He shows that senior entrepreneurs who sell the firm in the next period have an incentive to exert effort and hire young juniors who themselves also need to build a good reputation. Finally, Fleckinger (2014) considers collective reputation under Cournot competition where consumers only learn the average quality in the market. He studies the effect of the number of firms on welfare, and shows that quality is decreasing in the number of firms whereas quantity increases.

3 The Model

We consider a market with two firms that produce a vertically differentiated experience good, that can be of either good (G) or bad (B) quality, at zero cost. In every period, $t \in \{\dots, -1, 0, 1, \dots\}$, one short-lived consumer with unit demand arrives and is *randomly matched* with one of the firms.[5]

Each firm is competent (C) with probability μ or incompetent (I) with probability $1-\mu$, independently of the other firm. We interpret μ as the *baseline reputation* of firms in the economy. Firms' types are unobservable to consumers, but known to the firms. After being matched with a consumer, a competent firm can invest by incurring a cost of $c > 0$ in order to increase the probability that its product in that period is of good quality.[6] If a competent firm invests in period t, then its product has good quality (G) in the that period with probability π_H and bad quality (B) otherwise. If it does not invest, then the product has good quality with probability π_L, where $\pi_L < \pi_H$. An incompetent firm cannot invest and also produces good quality with probability π_L. Consumers do not observe firms' investment decisions, but they can observe the quality of goods produced in the last two periods through word-of-mouth or consumer reviews.[7] Using this information, they update their beliefs about the type of the firm they are matched with. After the investment decision

[5]Thus, we abstract away from (price) competition between the two firms.

[6]The qualitative analysis and results would not change if we assumed that investments are made prior to being matched. This specification simplifies the exposition of the firm's Bellman equation.

[7]Alternatively, one can think of a long-lived consumer with bounded memory.

firms make a take-it-or-leave-it offer to the consumer.[8] The consumer can either accept or reject the firm's offer and then she leaves the market.

Payoffs. We normalize the payoff of a consumer who does not buy the good to 0. If a consumer accepts a price p, she receives a payoff of $1-p$ if the good is of good quality, and $-p$ otherwise. A firm that sells in period t at price p_t receives a profit of $v_t = p_t - c$ at t if it invests at t and $v_t = p_t$ if it does not. Firms discount future payoffs by $\delta \in (0,1)$.

Information Structure. In every period, a consumer is assigned randomly to a firm. With a *collective brand*, consumers cannot distinguish between the identities of the two firms. This means that consumers obtain a signal about the brand in every period. If a firm maintains an *individual brand*, a consumer does not receive a signal about the firm that he is not matched with. Notice that the matching process ensures that the two firms sell the same expected quantities under the two regimes. We assume that firms know each other's types.

It follows that the set of relevant histories for an individual brand is $\mathcal{H}^{\text{ind}} = \{G, B, \varnothing\}^2$ where $\varnothing$ represents a failure to match. The set of relevant histories for a collective brand is $\mathcal{H}^{\text{col}} = \{G, B\}^2$. We denote the history at time t by $\mathbf{h}_t \equiv h_{t-2}h_{t-1} \in \mathcal{H}^b$ ($b \in \{\text{ind}, \text{col}\}$) where h_{t-n} denotes the quality of the good produced at $t-n$.

Equilibrium. We are interested in *stationary equilibria* in which strategies depend only on the relevant histories specified above. A stationary equilibrium is given by an investment and pricing strategy of firms, a purchasing strategy of consumers, and consumers' beliefs over the type of the firm they are matched with. For simplicity, we assume that consumers always purchase the good if it gives them a nonnegative expected payoff given their beliefs.

In the case of an *individual brand*, posterior beliefs given a history $\mathbf{h} \in \mathcal{H}^{\text{ind}}$ are given by a probability $\Pr^{\text{ind}}(C|\mathbf{h})$ that the firm that the consumer is matched with is competent. In the case of a *collective brand*, posterior beliefs are given by a probability distribution over the pair of types of the two firms. We denote the posterior belief that the two firms' types

[8] This assumption guarantees that all surplus goes to the firm in equilibrium, which simplifies the analysis. Firms must receive some surplus to have reputational concerns and for our results to hold.

are $s \in \{C, I\}^2$ given history $\mathbf{h} \in \mathcal{H}^{\text{col}}$ by $\eta_s(h)$ and so the posterior belief that the matched firm is competent given history $\mathbf{h}$ is $\Pr^{\text{col}}(C|\mathbf{h}) = \eta_{CC}(\mathbf{h}) + \frac{1}{2}(\eta_{CI}(\mathbf{h}) + \eta_{IC}(\mathbf{h}))$.

The reputation of a brand – both individual and collective – is given by the two posterior beliefs $\Pr^{\text{ind}}(C|\mathbf{h})$ and $\Pr^{\text{col}}(C|\mathbf{h})$, respectively, which can both be explicitly calculated using Bayes' Rule. In equilibrium, each player's strategy maximizes its payoffs given other players' strategies and beliefs. Posterior beliefs are derived from the realized histories and the firms' strategies by Bayes' rule whenever possible.

For most of the paper we focus our attention on the stationary equilibrium in which competent firms invest in quality whenever they are matched with a consumer, after each and every history. We call this the **reputational equilibrium**. This equilibrium is *socially optimal* if (and only if)

$$\Delta\pi \equiv \pi_H - \pi_L \geq c, \tag{1}$$

which we assume to be satisfied throughout the paper. This is also the brand profit-maximizing equilibrium in this case.

The game also has other stationary (and non-stationary) equilibria. For example, a "no investment" equilibrium, in which a competent firm never invests in quality, always exists. We discuss other stationary equilibria in Section 6 where we are concerned with endogenous brand formation.

4 Example

In order to intuitively understand the trade-offs present in our model, it is useful to think about the following illustrative example.[9] Consider two drivers, Adam and Brian, who work for a company that provides limousine services. In every period, a consumer who needs the service arrives and the company randomly assigns her to either Adam or Brian.

[9] We thank Robert Zeithammer for suggesting this example.

After the ride, the consumer posts a review about the quality of the ride on the company's website, which displays the last two reviews given by consumers. The company can decide as a policy to reveal or conceal the names of the drivers on the reviews. In the former policy, new consumers can check past records of individual drivers, so drivers are building their reputation individually (individual brand). In the latter, consumers cannot distinguish between two drivers' records, so they are building a collective reputation (collective brand).

Each driver's competency type is drawn independently so that he is competent with a probability μ. Only competent drivers can exert effort at a cost $c > 0$ to provide a good transportation service (G) with probability $\pi_H \in (0, 1)$. If they do not exert effort, they are indistinguishable from the incompetent drivers in that they always provide a bad transportation service (B), i.e., $\pi_L = 0$. Consumers receive a payoff of 1 from a good transportation service and 0 otherwise. We call this set-up, in which a good outcome reveals the driver's competence, the "exclusive knowledge" environment. Since the company makes a take-it-or-leave-it price offer to consumers, it can extract the entire consumer surplus by charging their willingness to pay:

$$p^b(\mathbf{h}) = \Pr^b(C|\mathbf{h})\pi_H + (1 - \Pr^b(C|\mathbf{h}))\pi_L \tag{2}$$

where $b \in \{\text{ind}, \text{col}\}$ denotes the company's policy on whether to reveal names of drivers, and $\mathrm{P}r^b(C|\mathbf{h})$ is the posterior probability that consumers assign to the driver being competent given history $\mathbf{h}$ in the reputational equilibrium. With exclusive knowledge we, thus have $p^b(\mathbf{h}) = \Pr^b(C|\mathbf{h})\pi_H$. For simplicity, we assume that π_H is very close to 1. As π_H increases to 1, beliefs tend to prices. That is, $\lim_{\pi_H \to 1} p^b(\mathbf{h}) = \lim_{\pi_H \to 1} \Pr^b(C|\mathbf{h})$. We list these beliefs in Table 1 below.

As demonstrated in Table 1, consumers' posterior beliefs are different for an individual and a collective brand. A consumer's belief about an individual brand reaches 1 for any histories that contain a G, and she pays the full price. In contrast, in a collective brand the

History $\mathbf{h}$	$p^{\text{ind}}(\mathbf{h})$	$p^{\text{col}}(\mathbf{h})$
GG	1	$\frac{3\mu+1}{2\mu+2}$
$G\varnothing, \varnothing G$	1	—
GB, BG	1	$\frac{1}{2}$
$B\varnothing, \varnothing B$	0	—
BB	0	$\frac{\mu(1-\mu)}{2\mu^2-6\mu+4}$

Table 1: Prices in an individual and collective brand as $\pi_H \nearrow 1$.

consumer cannot be sure that a good outcome G was generated by the firm it is matched with. Therefore, a consumer's belief is bounded away from 1 and 0 even after the best and worst outcomes, respectively.

To study when and how a collective brand provides stronger incentives to invest in quality, we examine the conditions under which it is possible to sustain the reputational equilibrium with individual and a collective reputation. Without loss of generality, we focus on Adam's incentives and assume that Adam is competent.

Suppose that Adam is visited in period t after the history $\mathbf{h}_t = h_{t-2}G$ where $h_{t-2} \in \{G, B\}$. Adam can either exert effort or not. Assume that Adam is an individual brand and he exerts effort. This investment in quality made in period t only affects payoffs in periods $t+1$ (short-run) and $t+2$ (long-run), because its outcome will be observed starting in period $t+1$ and forgotten by period $t+3$. Since $\pi_H \approx 1$, in period $t+1$ Adam's history will be $\mathbf{h}_{t+1} = GG$ almost surely, but he is visitied with only probability $\frac{1}{2}$. So, Adam's expected payoff in period $t+1$ (short term) is $\frac{1}{2}(p^{\text{ind}}(GG) - c)$. In period $t+2$, Adam is again visited with a probability $\frac{1}{2}$ and the history is either $\mathbf{h}_{t+2} = GG$ or $G\varnothing$, depending on whether Adam was visited in period $t+1$ or not. So, Adam's expected payoff in period $t+2$ (long term) is $\frac{1}{4}(p^{\text{ind}}(GG) - c) + \frac{1}{4}(p^{\text{ind}}(G\varnothing) - c)$. Plugging in the beliefs from Table 1, the expected payoff from investment is

$$\frac{\delta}{2} \cdot \big(p^{\text{ind}}(GG) - c\big) + \frac{\delta^2}{4} \cdot \big(p^{\text{ind}}(GG) - c\big) + \frac{\delta^2}{4} \cdot \big(p^{\text{ind}}(G\varnothing) - c\big) \quad = \quad \big(\delta + \delta^2\big) \cdot \frac{1-c}{2}.$$

If Adam does not exert effort in period t, then he saves the cost of effort at t at the

expense of a worse reputation in the subsequent two periods. Given the history $\mathbf{h}_t = h_{t-2}G$ and $\pi_L = 0$, $\mathbf{h}_{t+1} = GB$, and $\mathbf{h}_{t+2} = BG$ or $B\varnothing$, depending on whether Adam is visited in period $t+1$. Thus, the expected payoff from no investment is

$$\frac{\delta}{2} \cdot \left(p^{\text{ind}}(GB) - c\right) + \frac{\delta^2}{4}\left(p^{\text{ind}}(BG) - c\right) + \frac{\delta^2}{4}\left(p^{\text{ind}}(B\varnothing) - c\right) \quad = \quad \delta \cdot \frac{1-c}{2} + \delta^2 \cdot \frac{1-2c}{4}.$$

It follows that the *benefit* or *return from investment* in period t is

$$\left(\delta + \delta^2\right) \cdot \frac{1-c}{2} - \left[\delta \cdot \frac{1-c}{2} + \delta^2 \cdot \frac{1-2c}{4}\right] \quad = \quad \frac{\delta^2}{4}.$$

As long as this benefit is greater than the investment cost c, a deviation after the history $h_{t-2}G$ is not profitable.

Note that Adam's short-term payoffs are the same regardless whether Adam invests in period t or not. Thus, Adam has no short-run incentive to invest. This observation illustrates the severe short-run moral hazard problem that Adam faces when he attempts to establish an individual reputation; Adam has an incentive to exploit his current reputation when it is very good.

However, Adam has a long-run incentive to invest at t because such investment improves $\mathbf{h}_{t+2}$. If Adam knew that h_{t+1} would be G, Adam would have an incentive to deviate and save the investment cost at t, because he would be paid the full price in period $t+2$ anyway, independently of his investment at t. In other words, Adam has an incentive to free-ride on effort exerted by his future-self. But since Adam might not be visited in period $t+1$ while still being visited in period $t+2$, Adam has an incentive to invest in period t.

Similarly, it can be shown that the return on investment after a history $h_{-2}B$, with $h_{-2} \in \{G, B\}$, is $\frac{\delta}{2} + \frac{\delta^2}{4}$. Thus, the reputational equilibrium can be sustained if and only if the cost of investment is less than or equal to the minimum of return from the investment,

or

$$c \leq \min\left\{\frac{\delta^2}{4}, \frac{\delta}{2} + \frac{\delta^2}{4}\right\} = \frac{\delta^2}{4}.$$

Next, we consider the case in which Adam and Brian are evaluated anonymously, which corresponds to the case of a collective reputation. Suppose that both are competent. We calculate the maximum cost that still induces Adam to invest, given that Brian always exerts effort.

Suppose that assume that the brand has a history $\mathbf{h}_t = h_{t-2}G$ and that Adam is visited in period t. If Adam invests, he produces G with probability 1 and his expected payoff in period $t+1$ (short-run) is $\frac{1}{2} \cdot \big(p^{\text{col}}(GG) - c\big)$. In period $t+1$, since both drivers are competent, the brand produces G independently of who is hired to provides the service. Therefore, $\mathbf{h}_{t+2} = GG$, and Adam's expected payoff in that period (long-run) is again $\frac{1}{2} \cdot \big(p^{\text{col}}(GG) - c\big)$. In sum, Adam's expected payoff from investment at t is

$$\frac{\delta}{2}\left(p^{\text{col}}(GG) - c\right) + \frac{\delta^2}{2}\left(p^{\text{col}}(GG) - c\right) \quad = \quad \frac{1}{2}\left(\delta + \delta^2\right)\left(\frac{3\mu + 1}{2\mu + 2} - c\right).$$

If Adam does not invest in period t, then $\mathbf{h}_{t+1} = GB$ and $\mathbf{h}_{t+2} = BG$, which implies that his expected payoff is

$$\frac{\delta}{2} \cdot \left(p^{\text{col}}(GB) - c\right) + \frac{\delta^2}{2} \cdot \left(p^{\text{col}}(BG) - c\right) \quad = \quad \frac{1}{2}\left(\delta + \delta^2\right)\left(\frac{1}{2} - c\right).$$

It follows that Adam's return on investment is $\frac{1}{2}\left(\delta + \delta^2\right) \cdot \frac{\mu}{1+\mu}$.

In contrast to the case of individual reputation, the short-run return on investment does not vanish in the case of collective reputation. This is because as an anonymous driver who is a member of a group of anonymous drivers, Adam is never fully revealed to be competent. Even after the vest possible history GG, Adam has an incentive to exert effort in order to be paid a higher price in period $t+1$. Under collective reputation, if Adam invests in period

t, he can receive the price $p^{\text{col}}(GG) = \frac{3\mu+1}{2\mu+2}$ at $t+1$ instead of $p^{\text{col}}(GB) = \frac{1}{2}$, while under individual reputation $p^{\text{col}}(GG) = p^{\text{col}}(GB) = 1$.

Similarly, it is possible to show that the return on investment following a history $\mathbf{h}_t = h_{t-2}B$ with $h_{t-2} \in \{G, B\}$ is $\frac{\delta}{2} \cdot \frac{1-\mu}{2-\mu} + \frac{\delta^2}{2} \cdot \frac{\mu}{\mu+1}$. Thus, the reputational equilibrium exists if and only if

$$\begin{aligned} c &\leq \min\left\{\frac{1}{2}\left(\delta+\delta^2\right)\frac{\mu}{\mu+1}, \frac{\delta}{2}\cdot\frac{1-\mu}{2-\mu}+\frac{\delta^2}{2}\cdot\frac{\mu}{\mu+1}\right\} \\ &= \begin{cases} \frac{1}{2}\left(\delta+\delta^2\right)\frac{\mu}{\mu+1} & \text{if } \mu \leq \frac{1}{2} \\ \frac{\delta}{2}\cdot\frac{1-\mu}{2-\mu}+\frac{\delta^2}{2}\cdot\frac{\mu}{\mu+1} & \text{if } \frac{1}{2} \leq \mu \end{cases} \end{aligned}$$

Consequently, if μ is sufficiently large,[10] then a collective brand induces stronger incentives for effort than an individual brand in the sense that if the reputational equilibrium exists for an individual brand, it also exists for a collective brand as $\frac{\delta^2}{4} \leq c \leq \min\left\{\frac{1}{2}\left(\delta+\delta^2\right)\frac{\mu}{\mu+1}, \frac{\delta}{2}\cdot\frac{1-\mu}{2-\mu}+\frac{\delta^2}{2}\cdot\frac{\mu}{\mu+1}\right\}$. This is reversed if μ is not sufficiently large. Namely, in this case an individual reputation induces stronger incentives for effort than a collective reputation in the sense described above.

Importantly, as shown in the next section, the finding that a collective reputation may induce stronger incentives for effort than an individual reputation holds also in the case in which one of the drivers in the collective brand is incompetent.[11]

This example illustrates that a collective reputation may provide drivers with a more credible commitment to invest in quality. However, it should be noted that the interaction between short- and long-run incentives implies that a collective reputation induces relatively stronger incentives when δ is small, which is when short-run incentives are relatively more important which will be important for more general signal structures. This is the case when producers are not too patient or investments are only made infrequently.

[10] Sufficiently large so that $\frac{\delta^2}{4} \leq \min\left\{\frac{1}{2}\left(\delta+\delta^2\right)\frac{\mu}{\mu+1}, \frac{\delta}{2}\cdot\frac{1-\mu}{2-\mu}+\frac{\delta^2}{2}\cdot\frac{\mu}{\mu+1}\right\}$ or $\mu \geq \frac{\delta}{2+\delta} \leq \frac{1}{3}$.

[11] We do not consider the case in which both drivers are incompetent because in this case no driver can invest anyway.

5 Reputational Equilibrium

As illustrated in the example, the reputational equilibrium exists if and only if the cost of investment is less than a threshold that is equal to the benefits that are generated by the equilibrium. In this section, we compare the threshold for an individual and a collective brand and identify conditions under which a collective brand sustains a reputational equilibrium for a wider range of investment costs.

5.1 Individual Brand

In a reputational equilibrium the equilibrium price after history $\mathbf{h}_t = h_{t-2}h_{t-1} \in \mathcal{H}^{\text{ind}} = \{G, B, \varnothing\}^2$ is given by $p^{\text{ind}}(\mathbf{h}_t)$ as defined in (2).

Proposition 1. *The reputational equilibrium exists for an individual brand if and only if the cost of investment c satisfies*

$$c \leq \hat{\mathbf{c}}^{ind} \quad \equiv \quad \min_{h_{t-1} \in \{G,B,\varnothing\}} \bar{c}^{ind}(h_{t-1})$$

where $\bar{c}^{ind}(h_{t-1})$ denotes the expected benefit from investment given history $\mathbf{h}_t = h_{t-2}h_{t-1}$:

$$\begin{aligned} \bar{c}^{ind}(h_{t-1}) \equiv \quad & (\pi_H - \pi_L) \cdot \big[\tfrac{\delta}{2}\left(p^{ind}(h_{t-1}G) - p^{ind}(h_{t-1}B)\right) + \tfrac{\delta^2}{4}\big(\pi_H(p^{ind}(GG) - p^{ind}(BG)) + \\ & (1-\pi_H)(p^{ind}(GB) - p(BB)) + p^{ind}(G\varnothing) - p^{ind}(B\varnothing)\big)\big]. \end{aligned} \tag{3}$$

The stationary structure of the model allows us to express all the equilibrium probabilities $\Pr^{\text{ind}}(C|\mathbf{h}_t)$ and prices $p^{\text{ind}}(\mathbf{h}_t)$ in terms of the primitives of the model. For example, $p^{\text{ind}}(GG) = \frac{\mu\pi_H^3 + (1-\mu)\pi_L^3}{\mu\pi_H^2 + (1-\mu)\pi_L^2}$ because $\Pr^{\text{ind}}(C|GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}$. This implies that equation (3) above can be explicitly solved in terms of the parameters of the model alone. However, the solution is long and tedious, and conveys no important insight in itself so we do not provide it here.

Since investment should be the firm's optimal decision after all histories, the equilibrium exists if and only if the cost is less than the minimum of return on investment across all

possible histories. Given the most recent outcome h_{t-1} at time t, the benefit from investment, $\bar{c}^{\text{ind}}(h_{t-1})$, consists of a sum of price premiums that are obtained in the next two periods $t+1$ and $t+2$. The firm enjoys a positive return on investment at t only if it leads to a better outcome (i.e., G instead of B) and if it is visited again in at least one of the next two periods. Accordingly, the return on investment must be multiplied by $(\pi_H - \pi_L)$; the *short-run benefit* obtained in period $t+1$ must be multiplied by $\frac{\delta}{2}$; and the *long-run benefit* obtained in period $t+2$ must be multiplied by $\frac{\delta^2}{4}$. We also note from equation (3) that $\bar{c}^{\text{ind}}(h_{t-1})$ does not depend on the entire history $\mathbf{h}_t = h_{t-2}h_{t-1}$ because h_{t-2} will be forgotten by period $t+1$ and is thus irrelevant to the return from the investment.

In the short-run, the firm's investment in period t can improve $\mathbf{h}_{t+1} = h_{t-1}h_t$, which allows for a price premium $p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)$. However, if the firm's reputation at t is either very strong or very weak, then additional investment by the firm has only a small effect on its reputation, and yields only a small price premium. In this case, the firm's moral hazard problem is manifested through the incentive it has to exploit its current reputation.

In the long-run, an investment in period t can improve $\mathbf{h}_{t+2} = h_t h_{t+1}$, and the history h_{t-1} becomes irrelevant. Recall that under the reputational equilibrium, the firm is supposed to invest in period $t+1$, so that the realized outcome h_{t+1} is G, B, and $\varnothing$ with probabilities $\frac{\pi_H}{2}$, $\frac{1-\pi_H}{2}$, and $\frac{1}{2}$, respectively. Accordingly, the expected long-run benefit from investment at t is a weighted sum over price premiums of a form $p^{\text{ind}}(Gh_{t+1}) - p^{\text{ind}}(Bh_{t+1})$, where each term is weighted by the probability of h_{t+1}. In the reputational equilibrium, the firm expects additional investment at $t+1$, which makes h_{t+1} more likely to be G and so weakens the incentive that the firm has to exert effort at t. In other words, the firm has an incentive to free-ride on its future investment effort, which hurts its long-run incentives to invest at t.

Figure 1 depicts $\hat{\mathbf{c}}^{\text{ind}}$ as a function of the baseline reputation μ. The three dotted curves represent $\bar{c}^{\text{ind}}(h_{t-1})$ for $h_{t-1} \in \{G, B, \varnothing\}$, respectively, and the solid line represents $\hat{\mathbf{c}}^{\text{ind}}$. For any $\mu \in [0,1]$, the reputational equilibrium can be sustained for any cost $c \leq \hat{\mathbf{c}}^{\text{ind}}$. The threshold $\hat{\mathbf{c}}^{\text{ind}}$ converges to zero as μ tends to zero and one, which implies that the

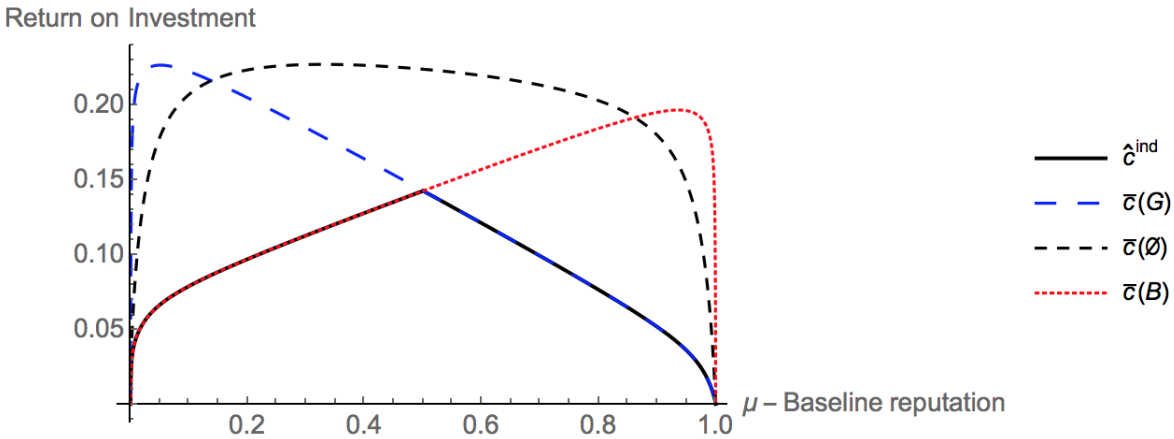


Figure 1: Return on Investment (ROI) after each history and the minimum $\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.4$

reputational equilibrium is harder to sustain in these cases. This is because for extreme values of μ, consumers' beliefs are not much affected by the brand's history and so the firm has little incentive to incur the investment cost. If μ is large, then the firm becomes complacent, or is tempted to "rest on its laurels." This complacency is the strongest after good outcomes, which generate the highest possible posterior beliefs. In this case, $\hat{\mathbf{c}}^{\text{ind}}$ is attained at $h_{t-1} = G$. If μ is small, then the firm becomes discouraged and gives up on improving its reputation. In this case, the firm's reputation plunges following a bad outcome and so $\hat{\mathbf{c}}^{\text{ind}}$ is attained at $h_{t-1} = B$.

The next proposition provides a characterization of the worst possible histories in terms of their effect on incentives.

Proposition 2. *If μ is sufficiently large, or*

$$\mu \geq \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H) + \pi_L(1-\pi_L)}$$

then $\hat{\mathbf{c}}^{ind} = \bar{c}^{ind}(G)$. Otherwise, $\hat{\mathbf{c}}^{ind} = \bar{c}^{ind}(B)$.

5.2 Collective Reputation

Consumers facing a collective brand cannot distinguish among the identities of individual firms within the brand. They care about the expected quality produced by the firm they are randomly matched with. Upon observing a history $\mathbf{h}_t \in \mathcal{H}^{\text{col}} = \{G, B\}^2$, a consumer forms beliefs about the quality of a random firm in the group, and is willing to pay $p^{\text{col}}(\mathbf{h}_t)$ given by (2).

Recall that firms' competency is known to firms, but not to consumers. Firms' payoffs depend on the type of the other member of the collective brand, denoted $\theta \in \{C, I\}$, and so do the returns from investment.

Proposition 3. *The reputational equilibrium exists for a collective brand if and only if the cost of investment c satisfies*

$$c \leq \hat{\mathbf{c}}^{col} \quad \equiv \quad \min_{h_{t-1} \in \{G,B\},\, \theta \in \{C,I\}} \bar{c}^{col}(h_{t-1}, \theta)$$

where

$$\begin{aligned} \bar{c}^{col}(h_{t-1}, \theta) \equiv \quad & \Delta\pi \cdot \big[\tfrac{\delta}{2}\left(p^{col}(h_{t-1}G) - p^{col}(h_{t-1}B)\right) + \\ & \tfrac{\delta^2}{4}\left((\pi_H + \pi(\theta))(p^{col}(GG) - p^{col}(BG)) + (2 - \pi_H - \pi(\theta))(p^{col}(GB) - p^{col}(BB))\right)\big], \end{aligned} \tag{4}$$

denotes the expected benefit from investment given history $\mathbf{h}_t = h_{t-2}h_{t-1}$ and where $\pi(\theta)$ denotes type θ's probability of producing high quality upon exerting effort, that is $\pi(\theta) = \pi_H$ if $\theta = C$ and $\pi(\theta) = \pi_L$ if $\theta = I$.

As in the case of an individual brand, equilibrium probabilities $\Pr^{\text{col}}(C|\mathbf{h}_t)$ and prices $p^{\text{col}}(\mathbf{h}_t)$ can be expressed in terms of the primitives of the model. However, since posterior beliefs are different from those for an individual brand, prices are also different. For example, $\Pr^{\text{col}}(C|GG) = \frac{\mu^2\pi_H^2 + \mu(1-\mu)(\frac{(\pi_H)^2}{4} + \frac{\pi_H\pi_L}{2} + \frac{(\pi_L)^2}{4})}{\mu^2\pi_H^2 + 2\mu(1-\mu)(\frac{(\pi_H)^2}{4} + \frac{\pi_H\pi_L}{2} + \frac{(\pi_L)^2}{4}) + (1-\mu)^2\pi_L^2}$. The expression for the price $p^{\text{col}}(GG)$ is even more complicated so we do not present it here.

As in the case of individual reputation, the short-run benefit from investment at t, ob-

tained in period $t+1$, is given by $p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)$. The long-run benefit from investment at t, obtained in period $t+2$, is a weighted sum of price premiums of the form $p^{\text{col}}(Gh_{t+1}) - p^{\text{col}}(Bh_{t+1})$. The realization of h_{t+1} depends on the effort provided by the brand in that period, which depends on which firm is visited and the type of the other firm. If $\theta = C$, then both members of the brand are competent and invest on the equilibrium path in period $t+1$ if matched. But, if $\theta = I$, then the brand invests only if the consumer at $t+1$ is matched with the competent firm. So, the probability that $h_{t+1} = G$ is $\frac{\pi_H + \pi(\theta)}{2}$ where $\pi(\theta) = \pi_H$ if $\theta = C$, and $\pi(\theta) = \pi_L$ otherwise.

A collective brand with two competent firms generates more good outcomes than an individual brand because the group invests in every period, whereas an individual brand does not invest when it is not matched with a consumer. A member of a collective brand is thus relatively more tempted to *free-ride* on the future investment of other members in a collective brand compared to and individual brand because today's deviation can be undone by tomorrow's effort. This consideration suggests that a collective brand has weaker long-run incentives to invest than an individual brand.

If μ is large, then the firm faces the commitment problem that is due to complacency as in the case of individual reputation. This problem is worst after the best possible history $h_{t-1} = G$ and when the firm expects more effort in the future, $\theta = C$. Thus, in this case $\hat{\mathbf{c}}^{\text{col}}$ is attained at $(h_{t-1}, \theta) = (G, C)$. If μ is small, then the commitment problem is due to discouragement. This problem is more severe following the worst possible history $h_{t-1} = B$ and when the firm expects less effort in the futre, $\theta = I$. Consequently, in this case $\hat{\mathbf{c}}^{\text{col}}$ is attained at $(h_{t-1}, \theta) = (B, I)$.

Proposition 4. *For μ close to 1, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(G, C)$, and for μ close to 0, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(B, I)$.*

Identification of the (h_{t-1}, θ) pair on which the threshold $\hat{\mathbf{c}}^{\text{col}}$ is attained is more complicated in the case of a collective compared to an individual reputation. The reason that this is so is demonstrated by Figure 2 below, which depicts $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ for the case in which $(h_{t-1}, \theta) = (G, C)$ and (B, I).

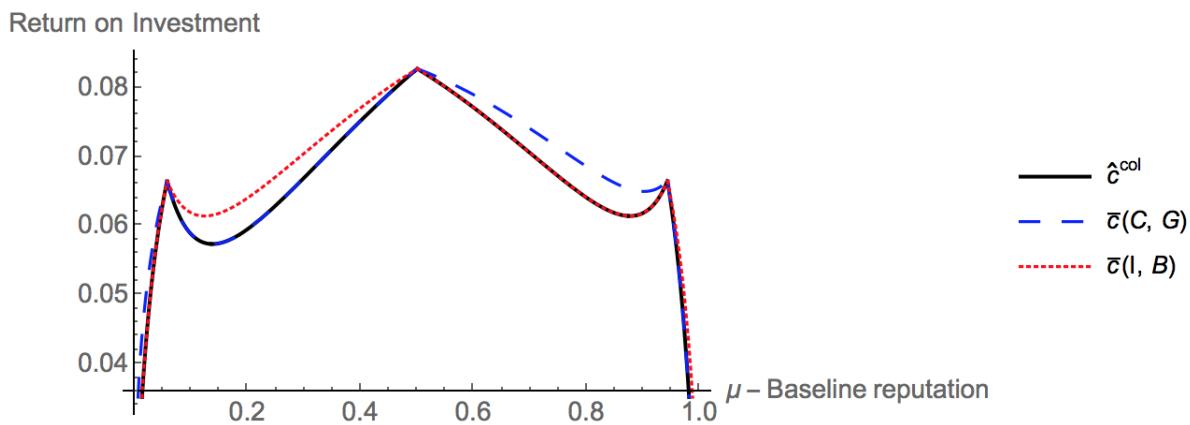


Figure 2: Return on Investment under (G, C) and (B, I)
$\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.4$

5.3 Comparing Individual and Collective Reputations

Having characterized the reputational equilibrium for an individual and a collective brand, we now compare the two types of branding. That is, we identify sufficient conditions for $\hat{\mathbf{c}}^{\text{ind}} < \hat{\mathbf{c}}^{\text{col}}$ to be satisfied. We focus our analysis on one special signal structure that is suggestive of a specific type of industry:

> **"Exclusive knowledge"** ($\pi_L \approx 0$): If π_L is very small, then it is nearly impossible to produce good quality without investment. In this case, an observation of good quality reveals that a firm is competent. This structure fits industries where competence represents a possession of a special technology or expertise that is required of producing high quality (e.g., for watches, automobiles, electronics, agriculture, etc.).

In the special case of exclusive knowledge, observation of a good outcome completely reveals the firm's type. This is bad for short-run incentives, especially in the case of individual reputation, because the short-run price premium $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) = 1 - 1 = 0$. The problem is mitigated for a collective brand because the revelation of one firm's type still

leaves some uncertainty with respect to the other firm's type. Since consumers do not know with which firm in the brand they are matched with, the short-run premium $p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)$ is strictly positive for all histories h_{t-1}.

Since short-run incentives for investment are stronger for a collective brand, a collective brand always induces stronger incentives for effort ($\hat{\mathbf{c}}^{\text{col}} > \hat{\mathbf{c}}^{\text{ind}}$) if δ is small enough. However, as noted above, an individual brand induces stronger long-run incentives. This is because an individual brand provides less future effort to free-ride on and because a firm in an individual brand may not have an opportunity to recover its damaged reputation after the production of bad quality. Therefore, δ cannot be too large for $\hat{\mathbf{c}}^{\text{col}} > \hat{\mathbf{c}}^{\text{ind}}$ to be true.

The advantage of a collective reputation is particularly pronounced if baseline beliefs μ are high. This is because if prior beliefs are close to 1, then the observation of good quality moves beliefs close to one (in the case of individual reputation, all the way up to one). Moreover, in the case of an individual reputation, beliefs do not shift downwards even if bad quality is observed following the observation of good quality. However, in the case of a collective reputation, beliefs can drop significantly following the observation of bad quality. This implies that the incentives to invest are much stronger in the case of a collective reputation compared to that of an individual reputation.

On the flip side, if μ is small then an individual brand is provided with very strong incentive to invest and improve its reputation because the production of just one product of good quality fully reveals the brand's competence. On the other hand, a collective brand's incentive declines as μ decreases because consumers' limited powers of observation limit the extent to which firms can improve their collective reputation by investing in quality.

The following proposition formalizes this intuition.

Proposition 5. *Suppose that π_L is sufficiently close to* 0.

1. *A collective brand sustains a reputational equilibrium for higher investment costs than an individual brand (i.e., $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$) if*

- *given a π_H, consumers' prior belief μ about the firm's type is sufficiently high and δ is not too large or*
- *π_H is sufficiently large and $\mu > \frac{1}{3}$.*

2. *An individual brand sustains a reputational equilibrium for higher investment costs than a collective brand ($\hat{\mathbf{c}}^{col} < \hat{\mathbf{c}}^{ind}$) for a fixed δ, π_H if μ is sufficiently low.*

3. *For any fixed $\pi_H \in (0, 1]$ and $\mu \in (0, 1)$, there exists a $\overline{\delta} \in [0, 1]$ such that $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$ for $\delta < \overline{\delta}$ and $\hat{\mathbf{c}}^{col} < \hat{\mathbf{c}}^{ind}$ for $\delta > \overline{\delta}$.*

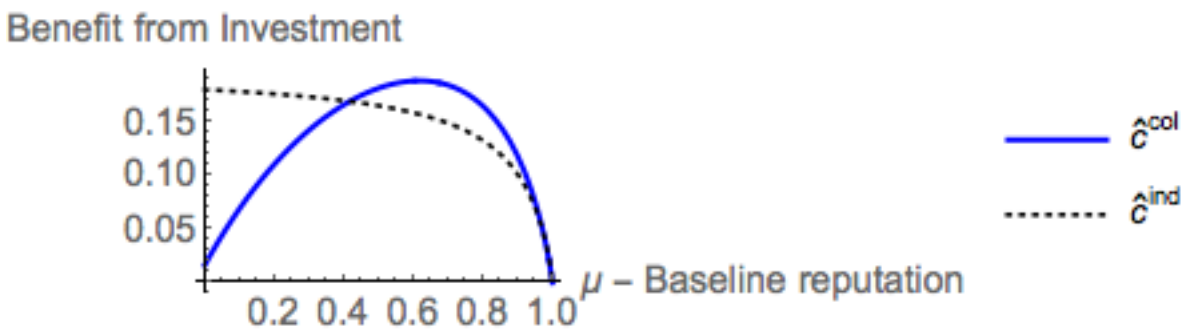


Figure 3: Comparison of Returns on Investment with $\pi_L = 0$
$\pi_H = 0.9,\ \delta = 0.9$

Figure 3 illustrates the benefit from investment. It shows that a collective brand dominates an individual brand for a wide range of priors μ. One caveat is that since $\delta = 0.9$ is large and π_H is hold fixed, for μ very close to 1, $\hat{\mathbf{c}}^{\text{col}} < \hat{\mathbf{c}}^{\text{ind}}$. The reason is that the benefit of collective brands is the most pronounced if π_H and μ grow simultaneously.

Remark 1. *(Quality control) The case in which π_H is close to 1, where a firm can almost always produce good quality if it invests, provides the mirror image of the case in which π_L is close to 0. It can be analyzed along similar lines and is therefore omitted. It fits industries where competence represents the ability to perform effective quality control, such as in the case of the manufacturing of generic products (e.g., nuts and bolts, widgets, etc.).*

6 Brand Formation

Until now, we treated the brand structure as exogenously given. This is a realistic assumption in some applications: A country might legally require labeling of the country of origin and the label of an appellation is often determined by the geographical location of the site of production. However, in other cases a firm may be able to choose whether to establish an individual reputation or to join a collective brand with other firms. This question is especially vexing in he case of a competent firm who considers whether to form a collective brand together with an incompetent firm.

For simplicity, in this section we restrict attention to the parameter values we identified in Proposition 5 where a collective brand indeed induces stronger investment incentives than an individual brand, i.e., we assume $\hat{\mathbf{c}}^{\text{ind}} < \hat{\mathbf{c}}^{\text{col}}$.

6.1 Stationary Equilibria for Individual Brands

We investigate which stationary equilibria exist for an individual brand when investment costs c are too high for a reputational equilibrium to exist. Recall that the set of relevant histories is given by $\mathcal{H}^{\text{ind}} = \{G, \varnothing, B\}^2$. A stationary equilibrium strategy specifies a mapping from the set of relevant histories into a decision of whether to invest or not. It can be characterized by a subset $\mathcal{S} \subset \mathcal{H}^{\text{ind}}$ of histories after which a competent firm invests.

As we noted in Proposition 1, the return from investment depends only on the outcome of the previous period, which implies that there can be at most $2^3 = 8$ candidates for stationary equilibria. The *no investment equilibrium*, where competent firms never invest, always exists. The reputational equilibrium is represented by the set $\mathcal{S} = \{G, B, \varnothing\}$ and the no investment equilibrium by $\mathcal{S} = \varnothing$.

Six other candidate stationary equilibria remain to be considered: $\mathcal{S} = \{\{G, \varnothing\}, \{G\}, \{\varnothing, B\}, \{B\}, \{G, B\}$, and $\{\varnothing\}\}$. [12] In the next proposition, we identify which stationary equilibria

[12] $\mathcal{S} = \varnothing$ represents the no investment equilibrium. $\mathcal{S} = \{\varnothing\}$ represents the stationary equilibrium in which competent firms invest only if they failed to match in the previous period.

exist when π_L is sufficiently close to zero (exclusive technology) and μ is close to one, which ensures that $\hat{\mathbf{c}}^{\text{ind}} < c < \hat{\mathbf{c}}^{\text{col}}$.

Proposition 6. *For π_L is sufficiently close to zero and μ is sufficiently close to one, the following is true*

1. *for $\hat{\mathbf{c}}^{col} > c > \hat{\mathbf{c}}^{ind}$, the only equilibrium that exists for an individual brand is the "no investment" equilibrium where a competent firm never invests.*[13]

2. *$c > \hat{\mathbf{c}}^{col}$, the only equilibrium that exists for a collective brand is the "no investment" equilibrium.*

To understand why no other stationary equilibria exist under the conditions specified in Proposition 3, we discuss here the reason why the two equilibria $\mathcal{S} = \{G, \varnothing\}$ and $\mathcal{S} = \{G\}$ fail to exist. The arguments for other equilibria are similar. For both equilibria, the firm's optimal decision following a bad outcome is to not invest. Knowing this, consumers pay a low price (equal to π_L) to a firm who produced a bad quality in the previous period. At the same time, a firm that just produced a good outcome is maximally rewarded with a price equal to π_H because it reveals the firm's competence. The large difference between the firm's payoff after bad and good outcomes implies that a firm would benefit from deviating and investing after a bad history because it would generate a higher payoff than non investment. Deterring this deviation requires that the cost of investment c is larger than $\hat{\mathbf{c}}^{ind}$, which is precluded by assumption.

6.2 Profits and Endogenous Brand Formation

Next, we compare the firm's expected payoff in each equilibrium. In any stationary equilibrium, the brand's history evolves according to the equilibrium investment strategy and

[13] If π_H is close to 1 (quality control) and μ is close to 0, then the equilibrium $\mathcal{S} = \{G, \varnothing\}$ exists if $\hat{\mathbf{c}}^{\text{ind}} < c < \hat{\mathbf{c}}^{\text{col}}$, which is the case if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$, and the equilibrium $\mathcal{S} = \{G\}$ exists if and only if $\delta > \frac{2\pi_L^2}{1+2\pi_L^2}$. This makes analysis of this case more involved than the case of exclusive knowledge, but it can nevertheless be analyzed along similar lines.

the outcome realizations. The firm's expected payoff under the no investment equilibrium is $\frac{\pi_L}{2}$ because the firm never invests, and sells at the price π_L in every period in which it is matched (with probability $\frac{1}{2}$). The firm's expected payoff in the reputational equilibrium is calculated as follows: because a competent firm invests after every history, the probability of outcomes G, $\varnothing$, and B is $\frac{\pi_H}{2}$, $\frac{1}{2}$, and $\frac{1-\pi_H}{2}$, respectively, if the firm is competent and $\frac{\pi_L}{2}$, $\frac{1}{2}$, and $\frac{1-\pi_L}{2}$, respectively, if the firm is incompetent.

Given these stationary probabilities it is possible to calculate the probabilities of different histories $\Pr^{ind}(\mathbf{h})$ and consumers' posterior beliefs $\Pr^{ind}(C|\mathbf{h})$ using Bayes' rule. The firm's one period profit following a history $\mathbf{h}$ is

$$\Pr^{ind}(C|\mathbf{h}) \cdot \pi_H + (1 - \Pr^{ind}(C|\mathbf{h})) \cdot \pi_L - c,$$

because under the reputational equilibrium, a competent firm always invests after any history and an incompetent firm never invests. The *average expected per-period profit* of the firm is then a sum of profits over all possible histories, each weighted according to its stationary probability. It is given by:

$$\Pi^{ind} = \sum_{h_1 \in \mathcal{S},\, h_2 \in \mathcal{H}} \Pr^{ind}(h_2 h_1) \cdot \big(\Pr^{ind}(C|h_2 h_1) \cdot \pi_H + (1 - \Pr^{ind}(C|h_2 h_1)) \cdot \pi_L - c\big)$$

The average profit for a collective brand under the reputational equilibrium is computed similarly. The price that the consumer pays after a history $h_2 h_1$, $p^{col}(h_2 h_1)$, is given by (2). The stationary distribution over histories depends on the type of the collective brand $\omega \in \{CC, CI, II\}$. The stationary probability $\Pr^{col}_{\omega}(h_2 h_1)$ of history $h_2 h_1$ for a brand ω can be calculated using the corresponding transition probabilities. For example, for a history $\mathbf{h} = GG$,

$$\Pr^{col}_{CC}(GG) = \pi_H^2, \quad \Pr^{col}_{CI}(GG) = \left(\tfrac{\pi_H + \pi_L}{2}\right)^2, \quad \Pr^{col}_{II}(GG) = \pi_L^2$$

Finally, the expected average profit of a collective brand of type ω is

$$\Pi_{\omega}^{\text{col}} = \sum_{\mathbf{h}} \Pr_{\omega}^{col}(\mathbf{h}) \cdot p(\mathbf{h}) - c.$$

Proposition 7. *The following is true for π_L sufficiently close to 0 (exclusive knowledge) and consumers' prior belief μ close to 1:*

1. *If $c \in (\hat{\mathbf{c}}^{ind}, \hat{\mathbf{c}}^{col})$, then a competent firm prefers forming a collective brand with another firm to establishing an individual brand, regardless of the type the other firm.*[14]

2. *If $c \in (0, \hat{\mathbf{c}}^{ind})$, the firm is instead better off with its own individual brand.*

3. *If $c \in (\hat{\mathbf{c}}^{col}, \infty)$, then the firm is indifferent.*

Proposition 7 shows that for industries that require exclusive knowledge and have high baseline reputation (μ close to 1), the commitment value of a collective brand can induce competent firms to want to brand with another firm regardless of its level of competence. This is the case for intermediate levels of costs of investment. For high or low investment costs, individual brands are always better. Thus, the following Corollary follows immediately.

Corollary 1. *In industries with exclusive knowledge (i.e., π_L sufficiently small) and high baseline reputation (i.e., μ sufficiently high), firms will always make the efficient branding decision, i.e., brand if and only if $c \in (\hat{c}^{ind}, \hat{c}^{col})$.*

Remark 2. *(Quality control) When π_H is close to 1 and the baseline reputation is low ($\mu \approx 0$), one can show that a competent firm always prefers not to brand. The reason is that firms now also face a lemons' problem: Since the probability of competence is small, the incentive*

[14] It can be shown that if π_H is close to 1 and μ is close to 0, then a competent firm always prefers to not form a collective brand. Intuitively, In a reputational equilibrium, consumers are willing to pay $\tilde{\mu} \cdot \pi_H + (1 - \tilde{\mu}) \cdot \pi_L$ if their posterior is $\tilde{\mu}$, and the firm incurs the cost c. If the firm is expected to not invest in equilibrium, consumers pay π_L. Then, roughly speaking, the firm prefers the reputational equilibrium if $\tilde{\mu} \cdot (\pi_H - \pi_L) > c$ "on average". This condition is violated for small $\tilde{\mu}$, in which case the firm is better off not investing. Thus, with endogenous brand formation, there is no investment in quality in the market, even though investment is socially optimal.

to commit to investing by forming a collective brand is too low even if $c \in (\hat{c}^{ind}, \hat{c}^{col})$*. Thus, in countries with relatively low baseline reputation (which could be developing countries) policy makers might want to force country of origin labeling.*

7 Extensions

7.1 T-period Memory

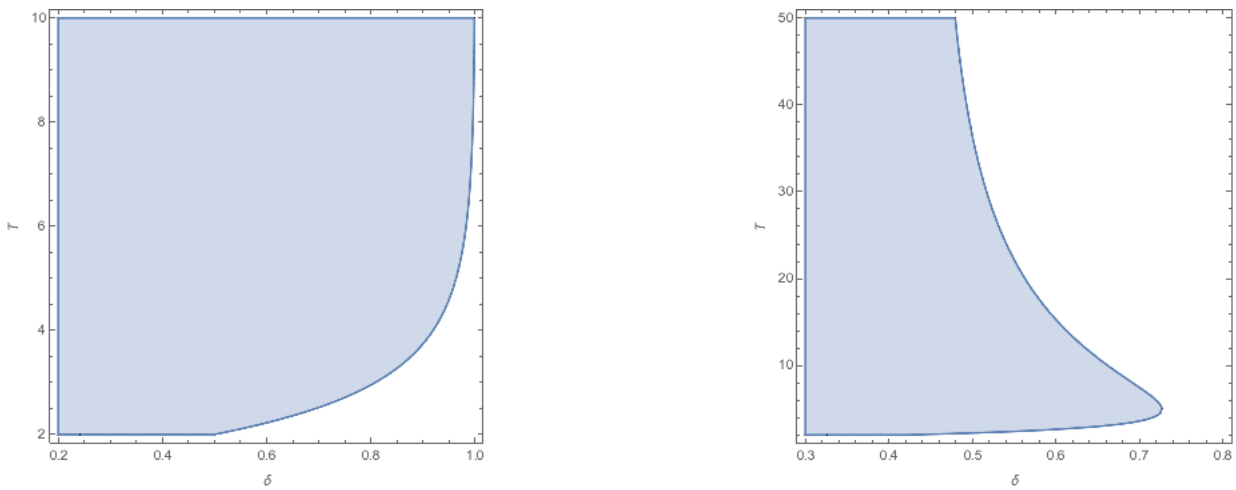
In this section, we show that our results extend to T- period memory of consumers. In fact, our results become stronger in the following sense: the range of discount factors δ for which a collective reputation provides a better commitment value than an individual reputation ($\hat{\mathbf{c}}^{\text{ind}} < \hat{\mathbf{c}}^{\text{col}}$) is non-empty for all T. Furthermore, in the limit as T tends to infinity, it is larger than in the case of a 2-period memory.

In general, with a longer memory, each single investment becomes less important. Thus, the benefit of investment decreases in T both in he case of individual and collective reputation. However, the benefit of investment in individual reputation is more adversely affected. The intuition is identical to that for the 2-period memory. With a longer memory, an individual brand can reach more extreme reputations following a sequence of good or bad outcomes, which worsens the associated moral hazard problem.

Here, we present only the main results and briefly compare them to the 2-period memory model described in Section 5. The detailed analysis of this case is relegated to Appendix B. The following proposition provides a generalization of Propositions 5 and **??**.

Proposition 8. *If* π_L *is sufficiently close to* 0 *(exclusive knowledge) and* μ *is sufficiently close to* 1*, then a collective brand sustains a reputational equilibrium for higher investment costs than an individual brand* ($\hat{\mathbf{c}}^{ind} < \hat{\mathbf{c}}^{col}$) *if either* $\delta < \frac{1}{2}$*, or* $\delta > \frac{1}{2}$ *and* $(2\delta)^T > \frac{\delta}{1-\delta}$.[15] *Moreover, the region of* δ *for which* $\hat{\mathbf{c}}^{ind} < \hat{\mathbf{c}}^{col}$ *increases monotonically in* T *and converges to* $[0, 1]$.

[15] A similar result holds if π_H is sufficiently close to 1 (quality control) and μ is sufficiently close to 0.



(a) $\pi_L = 0$ and $\mu = 1$

(b) $\pi_H = 1$, $\mu = 0$ and $\pi_L = 4/5$

Figure 4: Region in the T-δ space where $\hat{\mathbf{c}}^{\text{ind}} > \hat{\mathbf{c}}^{\text{col}}$.

Figure 4 exhibits the range of parameters for which a collective reputation induces stronger incentives to invest than an individual reputation. For the case of exclusive knowledge, a larger δ requires a correspondingly larger memory T for collective reputation to outperform an individual reputation.

8 Many Firms

With N firms, the type space of the firms is given by $\{C, I\}^n$. We keep assuming a 2 period memory and firms are drawn randomly out of a set of N, i.e., the histories are still given by $\mathcal{H}^{\text{ind}}$ and $\mathcal{H}^{\text{col}}$. The analysis of an individual brand is hence very similar to the main model.

We directly consider the limit $\pi_L \approx 0$, i.e., one good outcome G almost fully reveals that the firm is competent. Thus, $p^{\text{ind}}(\mathbf{h}) = \pi_H$ as long as $\mathbf{h}$ contains one G. The reputational equilibrium exists for an individual brand if and only if the cost of investment c satisfies

$$c \leq \hat{\mathbf{c}}^{\text{ind}}_{\mathbf{N}} \quad \equiv \quad \min_{h_{t-1} \in \{G, B, \varnothing\}} \bar{c}^{\text{ind}}_N(h_{t-1})$$

where $\bar{c}^{\text{ind}}(h_{t-1})$ denotes the expected benefit from investment given history $\mathbf{h}_t = h_{t-2}h_{t-1}$.

In the collective case, consumers facing a collective brand cannot distinguish between the identities of individual firms. They care about the expected quality of a randomly matched firm. Thus, the updating depends on the number of firms and it is weaker than with $N = 2$ because the consumer knows it is likely she does not observe any history of the firm it is matched with. All in all, Proposition 5 can be generalized.

Proposition 9. *Suppose that π_L is sufficiently close to 0. A collective brand sustains a reputational equilibrium for higher investment costs than an individual brand (i.e., $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$) if given a π_H, consumers' prior belief μ about the firm's type is sufficiently high and $\delta > 0$ is not too large. The cutoff at which δ, at which a collective brand is better for all π_L close to zero and μ close to one, is decreasing in the number of firms N.*

Thus, for exclusive knowledge industries with high base, a larger number of firms N decreases the benefit of collective reputation. This is in line with the intuition that the cost of free-riding is larger with a large number of firms.

9 Conclusion

We summarize the implications and interpretation of our results in the context of country of origin labelling. Recall that the prior belief μ can be interpreted as the *base reputation* of firms in a country.

Propositions 5 implies that in countries with strong base reputation (high μ) country of origin labeling contributes to social welfare by improving firms' ability to commit to invest in quality in industries with exclusive knowledge such as French wine, Swiss watches, German automobiles, Japanese electronics, US software, etc. In contrast, producers of generic products such as screws, basic clothes, etc., in such countries should advertise their own brand only. The exact opposite conclusion applies in countries with a weak base reputation (low μ). In such countries, social welfare is maximized when manufacturers of generic goods label

their country of origin while manufacturers of specialized goods avoid it.

These theoretical results are consistent with anecdotal evidence. For example the collective brand "Made in China" is advertised by subsuppliers on platforms such as 'Made-in-China.com'," while successful high-tech companies such as Huawei try to build their own brand names. On the other hand, German sub-suppliers of generics such as ThyssenKrupp count on their own brand reputation. This distinction is also consistent with the choice of the label "Made in Germany" versus "Made in Europe," which is supposedly indicative of a lower base reputation (lower μ).

Because firms in "quality control" industries may be reluctant to form a collective brand, the implementation of the optimal branding strategy might require some government regulation if the base reputation of firms is low. Indeed, the regulation of the labeling of country of origin is an important issue in many countries. The standard argument is that firms should be required to label their product with certain information in order to provide better consumer protection. The insights developed here suggest that the type of labeling, in particular the inclusion of country of origin, may affect the incentives of firms to invest in quality.

References

Bar-Isaac, Heski, "Something to prove: reputation in teams," *The RAND Journal of Economics*, 2007, *38* (2), 495–511.

Cabral, Luis MB, "Stretching firm and brand reputation," *RAND Journal of Economics*, 2000, pp. 658–673.

Choi, Jay Pil, "Brand extension as informational leverage," *The Review of Economic Studies*, 1998, *65* (4), 655–669.

Fishman, Arthur, Avi Simhon, Israel Finkelshtain, and Nira Yacouel, "The economics of collective brands," *Bar-Ilan University Department of Economics Research Paper*, 2014, (2010-11).

Fleckinger, Pierre, "Regulating Collective Reputation," 2014, p. 26.

Levin, Jonathan, "The dynamics of collective reputation," *The BE Journal of Theoretical Economics*, 2009, *9* (1).

Miklós-Thal, Jeanine, "Linking reputations through umbrella branding," *Quantitative Marketing and Economics*, 2012, *10* (3), 335–374.

Moorthy, Sridhar, "Can brand extension signal product quality?," *Marketing science*, 2012, *31* (5), 756–770.

Tirole, Jean, "A theory of collective reputations," *Research Papers in Economics University of Stockholm*, 1993, (9).

Wernerfelt, Birger, "Umbrella branding as a signal of new product quality: An example of signalling by posting a bond," *The RAND Journal of Economics*, 1988, pp. 458–466.

Zhang, Kaifu, "Breaking free of a stereotype: Should a domestic brand pretend to be a foreign one?," *Marketing Science*, 2015, *34* (4), 539–554.

A Appendix: Proofs

A.1 Proofs of Section 5

Proof. [Proposition 1] The posterior beliefs $\hat{\mu}^{\text{ind}}$ about the quality of the product after observing history $\mathbf{h}_t = h_{t-2}h_{t-1}$ are given by Bayes' rule:

$$\hat{\mu}^{\text{ind}}(GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}, \qquad \hat{\mu}^{\text{ind}}(GB) = \hat{\mu}^{\text{ind}}(BG) = \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)},$$

$$\hat{\mu}^{\text{ind}}(BB) = \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}, \qquad \hat{\mu}^{\text{ind}}(G\varnothing) = \hat{\mu}(\varnothing G) = \frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L}$$

$$\hat{\mu}^{\text{ind}}(\varnothing\varnothing) = \mu, \qquad \hat{\mu}^{\text{ind}}(B\varnothing) = \hat{\mu}^{\text{ind}}(\varnothing B) = \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}.$$

The reputational equilibrium exists if and only if a competent firm invests whenever visited following all histories, i.e.,

$$p^{\text{ind}}(h_{t-2}h_{t-1}) - c + \delta \cdot \big(\pi_H V(h_{t-1}G) + (1-\pi_H)V(h_{t-1}B)\big) \geq$$
$$p^{\text{ind}}(h_{t-2}h_{t-1}) + \delta \cdot \big(\pi_L V(h_{t-1}G) + (1-\pi_L)V(h_{t-1}B)\big)$$

which is equivalent to

$$c \leq \delta \cdot (\pi_H - \pi_L) \cdot (V(h_{t-1}G) - V(h_{t-1}B)).$$

Then, $V(h_{t-1}G)$ and $V(h_{t-1}B)$ can be written as

$$V(h_{t-1}G) = \frac{p^{\text{ind}}(h_{t-1}G) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(GG) + (1-\pi_H)V(GB) + V(G\varnothing)\big)$$
$$V(h_{t-1}B) = \frac{p^{\text{ind}}(h_{t-1}B) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(BG) + (1-\pi_H)V(BB) + V(B\varnothing)\big).$$

Then, the difference is

$$V(h_{t-1}G) - V(h_{t-1}B) = \frac{p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)}{2} + \frac{\delta}{2} \cdot \big(\pi_H \underbrace{(V(GG) - V(BG))}_{\frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB)}{2}}$$
$$+ (1-\pi_H) \underbrace{(V(GB) - V(BB))}_{\frac{p^{\text{ind}}(GB) - p^{\text{ind}}(BB)}{2}} + \underbrace{V(G\varnothing) - V(B\varnothing)}_{\frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}}\big).$$

□

Proof. [Proposition 2] First, note that

$$\begin{aligned}
p^{\text{ind}}(GG) - p^{\text{ind}}(GB) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H^2}{\mu \pi_H^2 + (1-\mu)\pi_L^2} - \frac{\mu \pi_H (1-\pi_H)}{\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L (1-\pi_L)} \right) \\
&= \frac{\mu(1-\mu)\pi_H \pi_L (\pi_H - \pi_L)^2}{\Pr(GG) \cdot \Pr(GB)}, \\
p^{\text{ind}}(GB) - p^{\text{ind}}(BB) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H (1-\pi_H)}{\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L (1-\pi_L)} - \frac{\mu (1-\pi_H)^2}{\mu (1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2} \right) \\
&= \frac{\mu(1-\mu)(1-\pi_H)(1-\pi_L)(\pi_H - \pi_L)^2}{\Pr(GB) \cdot \Pr(BB)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H}{\mu \pi_H + (1-\mu)\pi_L} - \frac{\mu (1-\pi_H)}{\mu (1-\pi_H) + (1-\mu)(1-\pi_L)} \right) \\
&= \frac{\mu(1-\mu)(\pi_H - \pi_L)^2}{\Pr(G) \cdot \Pr(B)} \\
&\geq \min\{p^{\text{ind}}(GG) - p^{\text{ind}}(GB),\ p^{\text{ind}}(GB) - p^{\text{ind}}(BB)\}.
\end{aligned}$$

Hence, the minimum is attained at $h_{-1} = G$ if and only if

$$\begin{aligned}
& \tfrac{\pi_H \pi_L}{\Pr(GG) \cdot \Pr(GB)} \leq \tfrac{(1-\pi_H)(1-\pi_L)}{\Pr(GB) \cdot \Pr(BB)} \\
\Leftrightarrow \quad & \Pr(BB) \cdot \pi_H \pi_L \leq \Pr(GG) \cdot (1-\pi_H)(1-\pi_L) \\
\Leftrightarrow \quad & \pi_H \pi_L (\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2) \leq (1-\pi_H)(1-\pi_L)(\mu \pi_H^2 + (1-\mu)\pi_L^2) \\
\Leftrightarrow \quad & \mu \pi_H (1-\pi_H) \geq (1-\mu)\pi_L (1-\pi_L)
\end{aligned}$$

This inequality holds if and only if $\mu \geq \bar{\mu} \equiv \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H)+\pi_L(1-\pi_L)}$. □

Proof. [Proposition 3] Let us denote by $V(h;\theta)$ the present discounted expected equilibrium profit of a competent firm when branding with a θ-type firm after history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$ at the beginning of the period before the consumer is assigned to either firm.

Then, a reputational equilibrium exists if and only if for all $\mathbf{h}_t, \theta$

$$p^{\text{col}}(h_{t-2}h_{t-1}) - c + \delta \cdot \big(\pi_H V(h_{t-1}G;\theta) + (1-\pi_H)V(h_{t-1}B;\theta)\big) \geq$$
$$p^{\text{col}}(h_{t-2}h_{t-1}) + \delta \cdot \big(\pi_L V(h_{t-1}G;\theta) + (1-\pi_L)V(h_{t-1}B;\theta)\big).$$

This is equivalent to

$$c \leq \overline{c}^{\text{col}}(h_{t-1}) \equiv \delta \cdot (\pi_H - \pi_L) \cdot (V(h_{t-1}G;\theta) - V(h_{t-1}B;\theta)).$$

First, note that for all $q_1, q_2, x \in \{G, B\}$, we have that $V(q_1x,\theta) - V(q_2x,\theta) = \frac{p^{\text{col}}(q_1x) - p^{\text{col}}(q_2x)}{2}$. Using this, we can calculate

$$\begin{aligned} V(h_{t-1}G;\theta) - V(h_{t-1}B;\theta) &= \frac{p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)}{2} \\ &+ \frac{\delta\pi_H}{2}\underbrace{(V(GG,\theta) - V(BG,\theta))}_{\frac{p^{\text{col}}(GG)-p^{\text{col}}(BG)}{2}} + \frac{\delta(1-\pi_H)}{2}\underbrace{(V(GB,\theta) - V(BB,\theta))}_{\frac{p^{\text{col}}(GB)-p^{\text{col}}(BB)}{2}} \\ &+ \frac{\delta\pi(\theta)}{2}(V(GG,\theta) - V(BG,\theta)) + \frac{\delta(1-\pi(\theta)))}{2}(V(GB,\theta) - V(BB,\theta)) \end{aligned}$$

where $\pi(\theta) = \pi_L$ if $\theta = I$ and π_H if $\theta = C$. □

Proof. [Proposition 4] As noted in Section 3, upon observing a history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$, a consumer places a probability $\eta_s(\mathbf{h}_t)$ on the group's type $s \in \{CC, CI, IC, II\}$. These beliefs are given

by:

$$
\begin{aligned}
\eta_{CC}(GG) &= \frac{\mu^2\pi_H^2}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\
\eta_{CI}(GG) &= \eta_{IC}(GG) \\
&= \frac{\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right)}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\
\eta_{II}(GG) &= 1 - \eta_{CC}(GG) - 2\eta_{CI}(GG), \\
\eta_{CC}(GB) &= \frac{\mu^2\pi_H(1-\pi_H)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\
\eta_{CI}(GB) &= \eta_{IC}(GB) \\
&= \frac{\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\
\eta_{II}(GB) &= 1 - \eta_{CC}(GB) - 2\eta_{CI}(GB), \\
\eta_{CC}(BB) &= \frac{\mu^2(1-\pi_H)^2}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\
\eta_{CI}(BB) &= \eta_{IC}(BB) \\
&= \frac{\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right)}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\
\eta_{II}(BB) &= 1 - \eta_{CC}(BB) - 2\eta_{CI}(BB).
\end{aligned}
$$

Then, the consumer's posterior belief is about the firm being competent is given by

$$\Pr(\mathbf{h}_t) = \eta_{CC}(\mathbf{h}_t) + \frac{1}{2}(\eta_{CI}(\mathbf{h}_t) + \eta_{IC}(\mathbf{h}_t))$$

and $p^{\text{col}}(\mathbf{h}_t) = (\pi_H - \pi_L)\Pr(\mathbf{h}_t) + \pi_L$. Thus, the price differentials are given by:

$$
\begin{aligned}
p^{\text{col}}(GG) - p^{\text{col}}(GB) &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)\right)}{4\cdot\Pr(GG)\cdot\Pr(GB)} \\
p^{\text{col}}(GB) - p^{\text{col}}(BB) &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)\right)}{4\cdot\Pr(GB)\cdot\Pr(BB)}.
\end{aligned}
$$

Thus, $p^{\text{col}}(GG) - p^{\text{col}}(GB) < p^{\text{col}}(GB) - p^{\text{col}}(BB)$ if and only if

$$\frac{\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)}{\Pr(GG)} < \frac{\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)}{\Pr(BB)}$$

Taking the limit $\mu \to 1$ on both sides, the inequality becomes

$$\frac{\pi_H(\pi_H+\pi_L)}{\pi_H^2} < \frac{(\pi_H-\pi_L)^2 - 2(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)}{(1-\pi_H)^2}.$$

This is equivalent to $\pi_L(1-\pi_H) < \pi_H(1-\pi_H)$, i.e., it is always satisfied. Similarly, for $\mu \to 0$, the inequality is equivalent to

$$\frac{\pi_L(\pi_H+\pi_L)}{\pi_L^2} < \frac{(1-\pi_L)(2-\pi_H-\pi_L)}{(1-\pi_L)^2}$$

which simplifies to $\pi_H < \pi_L$ which is never satisfied. Thus, by continuity $p^{\text{col}}(GG) - p^{\text{col}}(GB) < p^{\text{col}}(GB) - p^{\text{col}}(BB)$ for sufficiently large μ and $p^{\text{col}}(GG) - p^{\text{col}}(GB) > p^{\text{col}}(GB) - p^{\text{col}}(BB)$ for sufficiently small μ. The statement of the proposition follows from the definition of $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ in (4).

One can show that unlike in the independent branding case, as μ increases, the binding history changes from B to G, back to B and then to G, but it does not yield additional insights, so we omit the proof and statement. □

Proof. [Proposition 5] Let us assume that π_L is fixed and sufficiently small. It follows from Proposition 2 that for μ sufficiently large $\bar{c}^{\text{ind}}(G)$ determines the cutoff cost. Also, by Proposition 4, for sufficiently large μ, $\bar{c}^{\text{col}}(G;C)$ determines $\hat{\mathbf{c}}^{\text{col}}$. Thus, it suffices to compare $\hat{\mathbf{c}}^{\text{ind}} = \bar{c}^{\text{ind}}(G)$ and $\hat{\mathbf{c}}^{\text{col}} = \bar{c}^{\text{col}}(G;C)$.

First, for an individual brand,

$$\begin{aligned}
\lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(G) &= \delta \cdot \frac{\pi_H}{2} \lim_{\pi_L \to 0} \left(1 + \frac{\delta \pi_H}{2}\right) \big(\underbrace{p^{\text{ind}}(GG) - p^{\text{ind}}(GB)}_{\to 0}\big) \\
&\quad + \frac{\delta(1-\pi_H)}{2} \big(p^{\text{ind}}(GB) - p^{\text{ind}}(BB)\big) + \frac{\delta}{2}(p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)) \\
&= \delta^2 \cdot \frac{\pi_H^2}{2}(1-\mu)\left(\frac{1-\pi_H}{2} \cdot \frac{1}{1-\mu\pi_H(2-\pi_H)} + \frac{1}{2} \cdot \frac{1}{1-\mu\pi_H}\right).
\end{aligned}$$

Then, $\lim_{\mu \to 1} \frac{1}{1-\mu} \lim_{\pi_L \to L} \bar{c}^{\text{ind}}(G) = \frac{\delta^2 \pi_H^2}{2(1-\pi_H)}$.

For a collective brand,

$$\begin{aligned}
\lim_{\pi_L\to 0}\bar{c}^{\text{col}}(G;C) &= \delta\cdot\frac{\pi_H}{2}\lim_{\pi_L\to 0}(1+\delta\pi_H)(p^{\text{col}}(GG)-p^{\text{col}}(GB))+\delta(1-\pi_H)(p^{\text{col}}(GB)-p^{\text{col}}(BB))\\
&= \delta\cdot\frac{\pi_H}{2}(1+\delta\pi_H)\cdot\frac{(1-\mu)\mu\pi_H}{(1+\mu)(2-(1+\mu)\pi_H)}\\
&\quad +\delta^2\cdot\frac{\pi_H}{2}\cdot(1-\pi_H)\cdot\frac{(1-\mu)\pi_H\left(2-\pi_H(1+\mu(2-\mu\pi_H))\right)}{\left((1-\mu\pi_H)^2+\mu(1-\pi_H)^2+1-\mu\right)(2-(1+\mu)\pi_H)}
\end{aligned}$$

Thus, $\lim_{\mu\to 1}\frac{1}{1-\mu}\lim_{\pi_L\to 0}\bar{c}^{\text{col}}(G;C)=\frac{\delta}{2}\frac{\pi_H^2(1+2\delta)}{4(1-\pi_H)}$.

So, $\lim_{\mu\to 1}\frac{1}{1-\mu}\lim_{\pi_L\to 0}\bar{c}^{\text{col}}(G;C)>\lim_{\mu\to 1}\frac{1}{1-\mu}\lim_{\pi_L\to 0}\bar{c}^{\text{ind}}(G)$ if and only if $\frac{1}{2}>\delta$. Thus, as long as $\delta\leq\frac{1}{2}$ for sufficiently small π_L and μ sufficiently close to 1 the reputational equilibrium is easier to sustain in a collective brand than in an individual brand, i.e., $\hat{\mathbf{c}}^{\text{col}}\geq\hat{\mathbf{c}}^{\text{ind}}$. Moreover, $\lim_{\pi_H\to 1}(1-\pi_H)\lim_{\pi_L\to 0}\bar{c}^{\text{col}}(G;C)>\lim_{\pi_H\to 1}(1-\pi_H)\lim_{\pi_L\to 0}\bar{c}^{\text{ind}}(G)$ for all $\delta\in[0,1]$ if and only if $\mu>\frac{1}{3}$.

Note for general μ,π_H, a collective brand can be better for higher values of δ, namely as long as

$$\delta<\frac{\mu}{(\mu+1)\left(-\frac{\mu\pi_H}{(\mu+1)}+\frac{(2-\pi_H(1+\mu))(\pi_H(\mu(2\pi_H-3)-1)+2)}{2(1-\mu\pi_H)(\mu(\pi_H-2)\pi_H+1)}-\frac{(1-\pi_H)(\pi_H(\mu(\mu\pi_H-2)-1)+2)}{(\mu\pi_H(\mu\pi_H+\pi_H-4)+2)}\right)}\equiv\overline{\delta}$$

which is for example satisfied for any $\delta\leq 1$ for sufficient large π_H as we have seen in the introductory example..

If μ is close to 0, we get

$$\begin{aligned}
\lim_{\mu\to 0}\frac{1}{1-\mu}\lim_{\pi_L\to 0}\bar{c}^{\text{ind}}(G) &= \pi_H\frac{\delta(2-\pi_H)}{2},\text{ and}\\
\lim_{\mu\to 0}\frac{1}{1-\mu}\lim_{\pi_L\to 0}\bar{c}^{\text{col}}(G) &= \pi_H\frac{\delta(2-\pi_H)(1-\pi_H)}{2(2-\pi_H)}.
\end{aligned}$$

Clearly, $\delta\cdot\frac{2-\pi_H}{2}>\delta\cdot\frac{1-\pi_H}{2}$ for all $\pi_H\in(0,1)$. So, for a μ close 0, there is a π_L close to 0 such that $\hat{\mathbf{c}}^{\text{ind}}\geq\hat{\mathbf{c}}^{\text{col}}$. □

A.2 Proofs of Section 6

Proof. [Proposition 6]

I. Individual brand: For all equilibria other than the reputational equilibrium and no investment equilibrium, the firm sometimes invest and other times not. This implies the cost of investment cannot be too large to ensure that a competent firm wants to invest after histories in $\mathcal{S}$, i.e., $c < \overline{C}^{\mathcal{S}}$ and not too small to ensure that a competent firm does not want to invest after histories not in $\mathcal{S}$, i.e., $c > \underline{C}^{\mathcal{S}}$. In the following we calculate these bounds for the four candidate equilibria $\mathcal{S} \in \{\{G\}, \{G, \emptyset\}, \{B\}, \{B, \emptyset\}, \{G, B\}\}$. It turns out that the bounds are the same for $\{G\}$ and $\{G, \emptyset\}$, as well as for $\{B\}$ and $\{B, \emptyset\}$.

1. In an equilibrium with $\mathcal{S} \in \{\{G, \emptyset\}, \{G\}\}$, following a good history, the firm finds it optimal to invest in quality if and only if

$$p^{\text{ind}}(xG) - c + \delta \cdot \big(\pi_H V(GG) + (1-\pi_H)V(GB)\big) > p^{\text{ind}}(xG) + \delta \cdot \big(\pi_L V(GG) + (1-\pi_L)V(GB)\big).$$ [16]

As before, $V(h)$ for a history $h \in \{G, B, \varnothing\}^2$ is the equilibrium payoff at the beginning of the period before the consumer is assigned to either firm. The condition above is equivalent to

$$c < \delta \cdot (\pi_H - \pi_L) \cdot (V(GG) - V(GB)).$$

Then, $V(GG)$ and $V(GB)$ are

$$V(GG) = \frac{p^{\text{ind}}(GG) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(GG) + (1-\pi_H)V(GB) + V(G\varnothing)\big)$$
$$V(GB) = \frac{p^{\text{ind}}(GB)}{2} + \frac{\delta}{2} \cdot \big(\pi_L V(BG) + (1-\pi_L)V(BB) + V(B\varnothing)\big).$$

[16] The price $p^{\text{ind}}(h)$ here depends on the equilibrium strategy, so a more precise notation must include the equilibrium, such as $p^{\text{ind}}_{\mathcal{S}}(h)$. For simplicity, whenever we focus on a specific equilibrium, we omit the notation for the equilibrium and simply use $p^{\text{ind}}(h)$.

Then, the difference $V(GG) - V(GB) \equiv A$ is

$$A = \frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H(\underbrace{V(GG) - V(GB)}_{=A}) + \pi_L(\underbrace{V(GB) - V(BG)}_{=-B})$$
$$+ (1 - \pi_L)(\underbrace{V(GB) - V(BB)}_{=0}) + \underbrace{V(G\varnothing) - V(B\varnothing)}_{\frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}}\big).$$

The underbraces show simplification of expressions in the equation above. Let us denote $V(GB) - V(BG) = -B$. $V(GB) - V(BB)$ vanishes because both the period payoff and continuation payoffs are the same.[17] $V(G\varnothing) - V(B\varnothing) = p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)$ because the continuation payoff is the same.

The expression B can be obtained similarly by computing $V(BG)$ and $V(GB)$ (we computed this before) and then subracting.

$$V(BG) = \frac{p^{\text{ind}}(BG) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(GG) + (1 - \pi_H)V(GB) + V(G\varnothing)\big).$$

So, the difference B is

$$B = \frac{p^{\text{ind}}(BG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H(\underbrace{V(GG) - V(GB)}_{=A}) + \pi_L(\underbrace{V(GB) - V(BG)}_{=-B})$$
$$+ (1 - \pi_L)(\underbrace{V(GB) - V(BB)}_{=0}) + \underbrace{V(G\varnothing) - V(B\varnothing)}_{\frac{p(G\varnothing) - p(B\varnothing)}{2}}\big).$$

So, we can solve for simultaneous equations:

$$A = \frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H \cdot A - \pi_L \cdot B + \frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}\big)$$
$$B = \frac{p^{\text{ind}}(BG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H \cdot A - \pi_L \cdot B + \frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}\big).$$

[17] Consumers pay π_L upon observing both GB and BB, so the difference in period payoff vanishes, too.

Then, we find that

$$A = \frac{-2c + 2(p^{\text{ind}}(GG) - p^{\text{ind}}(GB)) + \delta(\pi_L \cdot (p^{\text{ind}}(GG) - p^{\text{ind}}(BG)) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))}{2(2 - \delta \cdot \Delta\pi)}.$$

Finally, we plug this into the initial condition for incentive compatibility and collect c on the same side:

$$\begin{aligned} c &< \frac{\delta(\pi_H - \pi_L)}{2} \cdot \left(p^{\text{ind}}(GG) - p^{\text{ind}}(GB) + \frac{\delta}{2}(\pi_L(p^{\text{ind}}(GG) - p^{\text{ind}}(BG)) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))\right) \\ &= \overline{C}^{\mathcal{S}}. \end{aligned}$$

Analogously, one can show that

$$\underline{C}^{\mathcal{S}} = \frac{\delta(\pi_H - \pi_L)}{2} \cdot \left(p^{\text{ind}}(BG) - p^{\text{ind}}(BB) + \frac{\delta}{2}(\pi_H \cdot (p^{\text{ind}}(GG) - p^{\text{ind}}(BG)) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))\right)$$

Note that these expressions are valid for both equilibria $\{G, \emptyset\}$ and $\{G\}$. However, p^{ind} will be different in the equilibria because the inferences made from observed histories varies.

(a) If a firm plays a $\{G, \emptyset\}$-equilibrium, then the transition matrix of the Markov chain governing the outcome is given by

$$\begin{pmatrix} \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \end{pmatrix}$$

Thus, the stationary distribution is given by $Pr(G|C) = \frac{\pi_H + \pi_L}{2(2-\pi_H+\pi_L)}$, $Pr(B|C) = \frac{1-\pi_H}{2-\pi_H+\pi_L}$, $Pr(\emptyset|C) = \frac{1}{2}$ if the firm is competent and $Pr(G|I) = \frac{\pi_L}{2}$, $Pr(B|I) = \frac{1-\pi_L}{2}$, $Pr(\emptyset|I) = \frac{1}{2}$. For example, $\mu(BG) = \frac{\mu\frac{1-\pi_H}{2-\pi_H+\pi_L}}{\mu\frac{1-\pi_H}{2-\pi_H+\pi_L} + (1-\mu)\frac{1-\pi_L}{2}}$ and $\mu(GG) =$

$\frac{\mu \frac{\pi_H+\pi_L}{2(2-\pi_H+\pi_L)}\pi_H}{\mu \frac{\pi_H+\pi_L}{2(2-\pi_H+\pi_L)}\pi_H+(1-\mu)\frac{\pi_L}{2}\pi_L}$. Then,

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\underline{C}^{\{G,\emptyset\}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0}\overline{C}^{\{G,\emptyset\}} = \frac{\delta\pi_H^2}{2} > 0.$$

Therefore, there is no $c > 0$ such that $\hat{c}^{\text{ind}} < c < \hat{c}^{\text{col}}$ for μ close to 1 that satisfy $\underline{C}^{\mathcal{S}} < c < \overline{C}^{\mathcal{S}}$ because $\lim_{\mu\to 1}\hat{c}^{\text{col}} = 0$.

Note that in the limit, prices simplify significantly and converge to either π_H or $\pi_L = 0$. Thus, we will skip the above steps for the other candidate equilibria. For the $\{G\}$-candidate equilibrium, we also get

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\underline{C}^{\{G\}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0}\overline{C}^{\{G\}} = \frac{\delta\pi_H^2}{2} > 0.$$

2. Likewise, for equilibria $\mathcal{S} = \{B, \varnothing\}$ or $\{B\}$, we find that

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\underline{C}^{\mathcal{S}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0}\overline{C}^{\mathcal{S}} = -\frac{\delta\pi_H^2}{2} < 0.$$

Therefore, these equilibria only exist when the investment cost is negative, and hence do not exist in our setup.

3. Then, the only two equilibria that need to be checked are $\mathcal{S} = \{G, B\}$ and $\{\varnothing\}$. These two equilibria demonstrate strategies non-monotonic in the firm's reputation in the sense that the firm takes the same action following a good and bad outcome, but a different one following an empty outcome.

   (a) In the candidate equilibrium $\mathcal{S} = \{G, B\}$, the firm must find it optimal to invest following a good or bad outcome, but not after an $\emptyset$-outcome, i.e., the equilibrium

exists if and only if

$$\frac{\pi_H - \pi_L}{2}(V^{\{G,B\}}(\varnothing G) - V^{\{G,B\}}(\varnothing B)) \leq c \leq \min_{x \in \{G,B\}} \frac{\pi_H - \pi_L}{2}(V^{\{G,B\}}(xG) - V^{\{G,B\}}(xB)).$$

Note, however, that the firm's investment decision only depends on the most recent outcome, i.e., the future payoffs in $V^{\{G,B\}}(yG)$ and $V^{\{G,B\}}(yB)$ are independent of $y \in \{G, \varnothing, B\}$. Therefore, the equilibrium can exist for some $c > 0$ if and only if

$$p^{\text{ind}}(\varnothing G) - p^{\text{ind}}(\varnothing B) < \min_{x \in \{G,B\}} p^{\text{ind}}(xG) - p^{\text{ind}}(xB).$$

If $\pi_L = 0$, the right-hand side is zero for $x = G$, as one good outcome reveals the firm to be competent. Therefore, the inequality cannot hold.

(b) We can similarly show the non-existence of the equilibrium $\mathcal{S} = \{\varnothing\}$: The equilibrium exists if and only if

$$\max_{x \in \{G,B\}} \frac{\pi_H - \pi_L}{2}(V^{\{\varnothing\}}(xG) - V^{\{\varnothing\}}(xB)) \leq c \leq \frac{\pi_H - \pi_L}{2}(V^{\{\varnothing\}}(\varnothing G) - V^{\{\varnothing\}}(\varnothing B)).$$

Therefore, the equilibrium can exist for some $c > 0$ if and only if $\max_{x \in \{G,B\}} p^{\mathcal{S}}(xG) - p^{\mathcal{S}}(xB) \leq p^{\mathcal{S}}(\varnothing G) - p^{\mathcal{S}}(\varnothing B)$. But both the left- and right-hand side are zero, as the firm does not invest following a good or bad outcome. Therefore, such equilibrium cannot exist if π_L is close to zero.

II. Collective brand: For a collective brand, competent firms must always play the same strategies in equilibrium (except for in knife-edge cases) because they either prefer to invest or not given equilibrium strategies. Thus, we focus on those equilibria.

In an equilibrium with $\mathcal{S} = \{G\}$, following a good history, a firm finds it optimal to

invest in quality if and only if

$$p^{\text{col}}(xG) - c + \delta \cdot \big(\pi_H V(GG) + (1-\pi_H)V(GB)\big) \geq p^{\text{col}}(xG) + \delta \cdot \big(\pi_L V(GG) + (1-\pi_L)V(GB)\big).$$

As before, for a history $h \in \{G, B\}^2$ $V(h)$ represents the equilibrium payoff at the beginning of the period before the consumer is assigned to either firm. This condition above is equivalent to $c \leq \delta \cdot (\pi_H - \pi_L) \cdot (V(GG) - V(GB))$. Similarly, after a bad history, a competent firm should not have an incentive to invest, i.e., the equilibirum exists if and only if

$$\delta \cdot (\pi_H - \pi_L) \cdot (V(BG) - V(BB)) \leq c \leq \delta \cdot (\pi_H - \pi_L) \cdot (V(GG) - V(GB)).$$

- Let us first calculate the bounds when both firms are competent C. Then, we can calculate for any $x \in \{G, B\}$

$$\begin{aligned} V(xG) &= \frac{p^{\text{col}}(xG) - c}{2} + \delta \cdot \big(\pi_H(V(GG) - V(GB)) + V(GB))\big) \\ V(xB) &= \frac{\pi_L}{2} + \delta \cdot \big(\pi_L V(BG) + (1-\pi_L)V(BB)\big). \end{aligned}$$

  and hence, for any $x, y \in \{G, B\}$

$$\begin{aligned} V(xG) - V(yB) = \frac{p^{\text{col}}(xG) - \pi_L - c}{2} &+ \delta \cdot \big(\pi_H(V(GG) - V(GB)) \\ &+ \pi_L(V(GB) - V(BG)) + (1-\pi_L)\underbrace{(V(GB) - V(BB))}_{=0}\big). \end{aligned}$$

  Thus, we can calculate $V(GG) - V(GB)$ and $V(GB) - V(BG)$ to be

$$\begin{aligned} V(GG) - V(GB) &= \frac{p^{\text{col}}(GG) - \pi_L - c + \delta\pi_L(p^{\text{col}}(GG) - p^{\text{col}}(BG))}{2(1 - \delta(\pi_H - \pi_L))} \\ V(BG) - V(GB) &= \frac{p^{\text{col}}(BG) - \pi_L - c + \delta\pi_H(p^{\text{col}}(GG) - p^{\text{col}}(BG))}{2(1 - \delta(\pi_H - \pi_L))} = V(BG) - V(BB). \end{aligned}$$

All in all, the equilibrium exists if and only if

$$\frac{\frac{\delta}{2}(\pi_H - \pi_L)}{1 - \frac{\delta}{2}(\pi_H - \pi_L)} \left(p^{\text{col}}(BG) - \pi_L + \delta\pi_H(p^{\text{col}}(GG) - p^{\text{col}}(BG))\right) \leq c \leq \\ \frac{\frac{\delta}{2}(\pi_H - \pi_L)}{1 - \frac{\delta}{2}(\pi_H - \pi_L)} \left((p^{\text{col}}(GG) - \pi_L) + \delta\pi_L(p^{\text{col}}(GG) - p^{\text{col}}(BG))\right)$$

- Let us next assume that the other firm is incompetent. In that case we have for any $x \in \{G, B\}$

$$V(xG) = \frac{p^{\text{col}}(xG) - c}{2} + \delta \cdot \left(\frac{\pi_H + \pi_L}{2}(V(GG) - V(GB)) + V(GB)\right) \\ V(xB) = \frac{\pi_L}{2} + \delta \cdot \left(\pi_L V(BG) + (1 - \pi_L)V(BB)\right).$$

and hence, for any $x, y \in \{G, B\}$

$$V(xG) - V(yB) = \frac{p^{\text{col}}(xG) - \pi_L - c}{2} + \delta \cdot \left(\frac{\pi_H + \pi_L}{2}(V(GG) - V(GB)) \right. \\ \left. + \pi_L(V(GB) - V(BG))\right).$$

All in all, the equilibrium exists if and only if

$$\frac{\frac{\delta}{2}\frac{\pi_H - \pi_L}{2}}{1 - \frac{\delta}{2}\frac{\pi_H - \pi_L}{2}} \left(p^{\text{col}}(BG) - \pi_L + \delta\frac{\pi_H + \pi_L}{2}(p^{\text{col}}(GG) - p^{\text{col}}(BG))\right) \leq c \leq \\ \frac{\frac{\delta}{2}\frac{\pi_H - \pi_L}{2}}{1 - \frac{\delta}{2}\frac{\pi_H - \pi_L}{2}} \left((p^{\text{col}}(GG) - \pi_L) + \delta\pi_L(p^{\text{col}}(GG) - p^{\text{col}}(BG))\right)$$

All in all, a $\{G\}$-equilibrium can be sustained if and only if

$$\underline{C}_{\text{col}}^{\{G\}} \equiv \frac{\frac{\delta}{2}(\pi_H - \pi_L)}{1 - \frac{\delta}{2}(\pi_H - \pi_L)} \left(p^{\text{col}}(BG) - \pi_L + \delta\pi_H(p^{\text{col}}(GG) - p^{\text{col}}(BG))\right) \leq c \leq \\ \frac{\frac{\delta}{2}\frac{\pi_H - \pi_L}{2}}{1 - \frac{\delta}{2}\frac{\pi_H - \pi_L}{2}} \left((p^{\text{col}}(GG) - \pi_L) + \delta\pi_L(p^{\text{col}}(GG) - p^{\text{col}}(BG))\right) \equiv \overline{C}_{\text{col}}^{\{G\}}$$

Taking $\pi_L \to 0$ yields

$$\frac{\frac{\delta}{2}\pi_H}{1-\frac{\delta}{2}\pi_H}\left(p^{\text{col}}(BG)+\delta\pi_H(p^{\text{col}}(GG)-p^{\text{col}}(BG))\right) \leq c \leq \frac{\frac{\delta}{2}\frac{\pi_H}{2}}{1-\frac{\delta}{2}\frac{\pi_H}{2}}p^{\text{col}}(GG) \tag{5}$$

In order to calculate the prices we need to calculate the stationary distribution of states given the transition matrix from the consumer's perspective. If both firms are competent it is given by

$$\begin{pmatrix} \pi_H & 1-\pi_H \\ \pi_L & 1-\pi_L \end{pmatrix}.$$

Thus, the stationary probability of being in state G is $Pr(G|CC) = \frac{\pi_L}{1-(\pi_H-\pi_L)}$ and the probability of being in state B is $Pr(B|CC) = \frac{1-\pi_H}{1-(\pi_H-\pi_L)}$. If one is competent and the other is incompetent, then it is given by

$$\begin{pmatrix} \frac{\pi_H+\pi_L}{2} & 1-\frac{\pi_H+\pi_L}{2} \\ \pi_L & 1-\pi_L \end{pmatrix}.$$

Thus, the stationary probability of being in state G is $Pr(G|CI) = \frac{\pi_L}{1-\frac{\pi_H-\pi_L}{2}}$ and the probability of being in state B is $Pr(B|CI) = \frac{1-\frac{\pi_H+\pi_L}{2}}{1-\frac{\pi_H-\pi_L}{2}}$. If both are incompetent, then $Pr(G|II) = \pi_L$ and the probability of being in state B is $Pr(B|II) = 1-\pi_L$. Note that as $\pi_L \to 0$, B always becomes an absorbing state. Hence, the after observing a history GG, a consumer updates his belief about facing a competent firm is

$$Pr(C|GG) = \frac{\mu^2\frac{\pi_L}{1-(\pi_H-\pi_L)}\pi_H + \mu(1-\mu)\frac{\pi_L}{1-\frac{\pi_H-\pi_L}{2}}\frac{\pi_H+\pi_L}{2}}{\mu^2\frac{\pi_L}{1-(\pi_H-\pi_L)}\pi_H + 2\mu(1-\mu)\frac{\pi_L}{1-\frac{\pi_H-\pi_L}{2}}\frac{\pi_H+\pi_L}{2} + (1-\mu)^2\pi_L^2}$$

and after observing a history BG, a consumer updates his belief about facing a competent firm is

$$Pr(C|BG) = \frac{\mu^2\frac{1-\pi_H}{1-(\pi_H-\pi_L)}\pi_L + \mu(1-\mu)\frac{1-\frac{\pi_H+\pi_L}{2}}{1-\frac{\pi_H-\pi_L}{2}}\pi_L}{\mu^2\frac{1-\pi_H}{1-(\pi_H-\pi_L)}\pi_L + 2\mu(1-\mu)\frac{1-\frac{\pi_H+\pi_L}{2}}{1-\frac{\pi_H-\pi_L}{2}}\pi_L + (1-\mu)^2\pi_L^2}$$

Thus, as $\pi_L \to 0$ (5) converges to

$$\lim_{\pi_L \to 0} \underline{C}_{\text{col}}^{\{G\}} = \frac{\frac{\delta}{2}\pi_H}{1-\frac{\delta}{2}\pi_H}\pi_H \mu \left(1 + \delta \pi_H \left(\frac{\mu \frac{\pi_H}{1-\pi_H} + (1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}}{\mu \frac{\pi_H}{1-\pi_H} + 2(1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}} - \frac{1}{2-\mu}\right)\right) \leq c$$

$$\leq \lim_{\pi_L \to 0} \overline{C}_{\text{col}}^{\{G\}} = \frac{\frac{\delta}{2}\frac{\pi_H}{2}}{1-\frac{\delta}{2}\frac{\pi_H}{2}}\pi_H \mu \frac{\mu \frac{\pi_H}{1-\pi_H} + (1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}}{\mu \frac{\pi_H}{1-\pi_H} + 2(1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}}.$$

Then, as $\mu \to 1$ we can write

$$\lim_{\mu \to 1} \frac{1}{\mu} \lim_{\pi_L \to 0} \underline{C}_{\text{col}}^{\{G\}} = \frac{\frac{\delta}{2}\pi_H}{1-\frac{\delta}{2}\pi_H}\pi_H \quad > \quad \frac{\frac{\delta}{2}\frac{\pi_H}{2}}{1-\frac{\delta}{2}\frac{\pi_H}{2}}\pi_H = \lim_{\mu \to 1} \frac{1}{\mu} \lim_{\pi_L \to 0} \overline{C}_{\text{col}}^{\{G\}}.$$

Thus, as for large μ, there is no $c > 0$ such that a $\{G\}$-equilibrium exists. Similarly, one can show that no $\{B\}-$equilibrium can exist for sufficiently large μ and π_L close to zero. □

Proof. [Proposition 7] For $c \in (\hat{\mathbf{c}}^{\text{ind}}, \hat{\mathbf{c}}^{\text{col}})$, the only equilibrium for an individual brand is the "no investment" equilibrium by Proposition 6. In this equilibrium, its average profits are given by $\lim_{\mu \to 1} \lim_{\pi_L \to 0} \Pi^{\text{ind}} \approx \pi_L \approx 0$. In a collective brand, regardless of the other firm's competency, the firm's average profit in a reputation equlibrium is given by $\lim_{\mu \to 1} \lim_{\pi_L \to 0} \Pi^{\text{col}} = \pi_H - c$. Therefore, the firm always prefers branding with another firm to staying alone as long as $c < \pi_H$ with is the case by assumption.

For $c \in (0, \min\{\hat{\mathbf{c}}^{\text{ind}}, \hat{\mathbf{c}}^{\text{col}}\})$, the reputational equilibrium exists for an individual and collective brand. Thus, after any history, consumers expect competent firms to invest, but the belief updating after a particular history is different. An individual firm makes an average profit of

$$\begin{aligned}\Pi^{\text{ind}} \quad = \quad & 0.25 \cdot \big(\pi_H^2 p^{\text{ind}}(GG) + 2 \cdot \pi_H(1-\pi_H)p^{\text{ind}}(GB) + (1-\pi_H)^2 p^{\text{ind}}(BB) + \\ & 2 \cdot \pi_H p^{\text{ind}}(G\emptyset) + 2 \cdot (1-\pi_H)p^{\text{ind}}(B\emptyset) + p^{\text{ind}}(\emptyset\emptyset)\big) - c\end{aligned}$$

A competent firm forms a brand with another competent firm makes an average profit of

$$\Pi^{\text{col}} = \pi_H^2 p^{\text{col}}(GG) + \pi_H(1-\pi_H)p^{\text{col}}(GB) + \pi_H(1-\pi_H)p^{\text{col}}(BG) + (1-\pi_H)^2 p^{\text{col}}(BB) - c.$$

Then, as $\mu \to 1$ and $\pi_L \to 0$ the difference in average profits satisfies

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Pi^{\text{ind}} - \Pi^{\text{col}}}{(1-\mu)^2} = \frac{\pi_H^3(\pi_H(\pi_H((0.125\pi_H - 0.5)\pi_H + 0.75) - 0.5) + 0.125)}{(1-\pi_H)^6} > 0.$$

Thus, for large μ, a firm always prefers to stay alone to branding with another firm. Note that branding with an incompetent firm is always less attractive than branding with a competent firm.

When $c > \hat{c}^{\text{col}} > \hat{c}^{\text{ind}}$, only the no-investment equilibrium exist for a collective brand, as well as an individual brand. Thus, the average profit in both scenarios is $\pi_L = 0$. □

A.3 Proof of Section 7.1

Proof. [Proof of Proposition 8] For the good news case with $\pi_L \approx 0$, we compare the cutoff levels we obtained by taking limit of μ to 1. $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$ in this region if $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta Z_C^{\text{col}}(G^{T-1})}{1-\mu} > \lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta Z^{\text{ind}}(G^{T-1})}{1-\mu}$.

$$\frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta} > \frac{\pi_H}{2(1-\pi_H)} \cdot \delta^{T-1}. \tag{6}$$

This holds if and only if either $\delta \leq \frac{1}{2}$, or $\delta > \frac{1}{2}$ and $(2\delta)^T > \frac{\delta}{1-\delta}$. Because $(2\delta)^T$ is increasing in T for $\delta > \frac{1}{2}$, there is a large enough $\bar{T}(\delta)$ such that for all $T > \bar{T}(\delta)$, (17) holds. This implies that a longer memory expands the region of δ under which a collective brand sustains the reputational equilibrium better. For example, if $T = 2$, it is supported for $\delta \in (0, \frac{1}{2})$, if $T = 3$, $\delta \in (0, \frac{1+\sqrt{5}}{4} \approx 0.809)$, and if $T = 4$, $\delta \in (0, 0.919)$.

The rest follows immediately from the equation (17). □

A.4 Proof of Section 8

Proof. [Proof of Proposition 3] Note that the history that must minimizes the benefit from investment is $h_{t-1} = GG$ and

$$\bar{c}_N^{\text{ind}}(GG) \equiv \pi_H^2 \frac{\delta^2}{N^2}\left[(1-\pi_H)\frac{1-\mu}{\mu(1-\pi_H)^2+1-\mu} + (N-1)\cdot\frac{1-\mu}{\mu(1-\pi_H)+1-\mu}\right].$$

Thus, we can write

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\bar{c}_N^{\text{ind}}(GG)}{1-\mu} = \frac{\pi_H^2\cdot\delta^2}{N}\frac{1}{(1-\pi_H)}$$

In the collective case, for $\pi_L \approx 0$ the probability of facing a C-firm after a history h with u G-observations and $2-u$ B-observations simplifies to

$$\lim_{\pi_L\to 0}\Pr^{\text{col}}(C|h) = \frac{\sum_{i=1}^N \binom{N}{i}\mu^i(1-\mu)^{N-i}\pi_H^u \sum_{v=u}^2 \binom{2-u}{v-u}\left(\frac{i}{N}\right)^{v+1}\left(\frac{N-i}{N}\right)^{2-v}(1-\pi_H)^{v-u}}{\sum_{i=0}^N \binom{N}{i}\mu^i(1-\mu)^{N-i}\pi_H^u \sum_{v=u}^2 \binom{2-u}{v-u}\left(\frac{i}{N}\right)^{v}\left(\frac{N-i}{N}\right)^{2-v}(1-\pi_H)^{v-u}}$$

Now we can calculate the price differences:

$$\lim_{\pi_L\to 0}\Pr^{\text{col}}(C|GG) - \Pr^{\text{col}}(C|GB) =$$

$$\sum_{i=1}^N \binom{N}{i}\mu^i(1-\mu)^{N-i}\sum_{j=1}^N \binom{N}{j}\mu^j(1-\mu)^{N-j}$$

$$\frac{\pi_H^2\left(\frac{i}{N}\right)^3\pi_H\left(\frac{j}{N}\left(\frac{N-j}{N}\right)+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)-\pi_H\left(\left(\frac{i}{N}\right)^2\left(\frac{N-i}{N}\right)+\left(\frac{i}{N}\right)^3(1-\pi_H)\right)\pi_H^2\left(\frac{j}{N}\right)^2}{\left(\sum_{j=1}^N\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H^2\left(\frac{j}{N}\right)^2\right)\left(\sum_{j=1}^N\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H\left(\frac{j}{N}\left(\frac{N-j}{N}\right)+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)\right)} =$$

$$\sum_{i=1}^N \binom{N}{i}\mu^i(1-\mu)^{N-i}\sum_{j=1}^N \binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H^3\left(\frac{i}{N}\right)^2\frac{j}{N}$$

$$\frac{\left(\frac{i}{N}\left(\frac{N-j}{N}\right)+\frac{i}{N}\frac{j}{N}(1-\pi_H)\right)-\left(\frac{j}{N}\left(\frac{N-i}{N}\right)+\frac{i}{N}\frac{j}{N}(1-\pi_H)\right)}{\left(\sum_{j=1}^N\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H^2\left(\frac{j}{N}\right)^2\right)\left(\sum_{j=1}^N\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H\left(\frac{j}{N}\left(\frac{N-j}{N}\right)+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)\right)} =$$

$$\sum_{i=1}^N\sum_{j=1}^N \binom{N}{j}\binom{N}{i}\mu^{i+j}(1-\mu)^{2N-i-j}\pi_H^3\left(\frac{i}{N}\right)^2\frac{j}{N}\frac{\frac{i}{N}-\frac{j}{N}}{Pr^{\text{col}}(GG)Pr^{\text{col}}(GB)}$$

and consequently

$$\lim_{\mu\to 1}\frac{1}{1-\mu}\lim_{\pi_L\to 0}(\Pr^{\text{col}}(C|GG)-\Pr^{\text{col}}(C|GB)) =$$
$$\pi_H^3\frac{N-1}{N^2}\frac{1}{\pi_H^3(1-\pi_H)} = \frac{N-1}{N^2(1-\pi_H)}$$

Similarly,

$$\lim_{\pi_L\to 0}(\Pr^{\text{col}}(C|GB)-\Pr^{\text{col}}(C|BB)) =$$
$$\sum_{i=1}^{N}\binom{N}{i}\sum_{j=0}^{N}\binom{N}{j}\mu^{i+j}(1-\mu)^{2N-i-j}\pi_H\frac{i}{N}$$
$$\frac{\left(\left(\frac{i}{N}\right)\frac{N-i}{N}+\left(\frac{i}{N}\right)^2(1-\pi_H)\right)\left(\left(\frac{N-j}{N}\right)^2+2\left(\frac{j}{N}\right)\frac{N-j}{N}(1-\pi_H)+\left(\frac{j}{N}\right)^2(1-\pi_H)^2\right)}{\Pr^{\text{col}}(GB)\Pr^{\text{col}}(BB)}$$
$$-\frac{\left(\left(\frac{j}{N}\right)\frac{N-j}{N}+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)\left(\left(\frac{N-i}{N}\right)^2+2\left(\frac{i}{N}\right)\frac{N-i}{N}(1-\pi_H)+\left(\frac{i}{N}\right)^2(1-\pi_H)^2\right)}{\Pr^{\text{col}}(GB)\Pr^{\text{col}}(BB)}$$

and consequently

$$\lim_{\mu\to 1}\frac{1}{1-\mu}\lim_{\pi_L\to 0}(\Pr^{\text{col}}(C|GB)-\Pr^{\text{col}}(C|BB)) = \frac{N+\pi_H-N\pi_H}{N^2(1-\pi_H)^2}$$

Thus, the cutoff for $\theta = C$ and $h = GG$ can be written as

$$\begin{aligned}\lim_{\mu\to 1}\frac{1}{1-\mu}\bar{c}^{\text{col}}(GG,C) &= \pi_H^2\left[\left(\frac{\delta}{N}+\frac{\delta^2}{N}\pi_H\right)\frac{N-1}{N^2(1-\pi_H)}+\frac{\delta^2}{N}(1-\pi_H)\frac{N-(N-1)\pi_H}{N^2\pi_H(1-\pi_H)^2}\right]\\ &= \frac{\delta(N-1+\delta N)\pi_H^2}{N^3(1-\pi_H)}\end{aligned}$$

Thus, we can write

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{1}{1-\mu}(\bar{c}^{\text{col}}(GG,C)-\bar{c}^{\text{ind}}(G)) = \pi_H^2\frac{\delta}{N}\left[\frac{N-1+\delta N}{N^2(1-\pi_H)}-\frac{\delta}{1-\pi_H}\right] = \frac{\delta\pi_H^2\left(2(N-1)-\delta N(N^2-2)\right)}{2N^3(1-\pi_H)}$$

Which is positive for $\delta < \frac{2(N-1)}{N(N^2-2)}$.

□

B Appendix: $T-$Period Memory

In this section, we extend our analysis to a T-period memory for $T > 2$. With a $T-$period memory, a relevant history at period t is of the form $\mathbf{h}_t \in \mathcal{H}^{\text{ind}} := \{G, \varnothing, B\}^T$ for an individual brand and $\mathbf{h}_t \in \mathcal{H}^{\text{col}} := \{G, B\}^T$ for a collective brand. The history consists of outcomes produced in the previous T periods, $\mathbf{h}_t = h_{t-T}h_{t-T+1}\cdots h_{t-1}$. As time proceeds, consumers' new history consists of the most recent outcomes from $\mathbf{h}_t$ and new outcomes. Let us denote the n most recent outcomes by $\mathbf{h}_t^n = h_{t-n}\cdots h_{t-1}$ for any $1 \leq n \leq T$.

As in Section 5, we start by finding conditions under which the reputational equilibrium exists for an individual and a collective brand. Then, we compare the respective parameter regions to find where the equilibrium exists under a collective, but not under an individual brand. The analysis is similar to that in Section 5, so to avoid redundancy, we omit repetitive details.

B.1 Individual brand

In a reputational equilibrium, a competent firm must find it optimal to invest after anyl history. To rule out profitable deviations, we consider the firm's investment decision at period t (also often referred as today) when the firm will invest whenever visited in the

future. By investing, it can add a $h_t = G$ to the history $\mathbf{h}_t$ with a greater probability, which will be remembered in the next T periods. $k+1$ periods after period t, consumers would have forgotten the $k+1$ oldest outcomes, and $k+1$ new outcomes are added to the relevant history

$$\mathbf{h}_{t+k+1} = \mathbf{h}_t^{T-k-1} \underbrace{h_t h_{t+1} \cdots h_{t+k}}_{\text{new outcomes}} = \mathbf{h}_t^{T-k} h_t \mathbf{h}_{t+k+1}^{k-1}.$$

The new outcomes are denoted by $h_t \mathbf{h}_{t+k+1}^k$, where h_t is the result of the focal investment decision. To simplify the notation and to distinguish the known (old) outcomes and those to be realized, we denote future outcomes $\mathbf{h}_{t+k+1}^k$. Then, conditional on realizing the future outcomes $\mathbf{f}$, the benefit of investing in period t comes from a probabilistic improvement in the history from $\mathbf{h}_t^{T-k-1} B \mathbf{h}_{t+k+1}^k$ to $\mathbf{h}_t^{T-k-1} G \mathbf{h}_{t+k+1}^k$. This allows the firm to receive a higher price $p^{\text{ind}}(\mathbf{h}_t^{T-k-1} G \mathbf{h}_{t+k+1}^k) - p^{\text{ind}}(\mathbf{h}_t^{T-k-1} B \mathbf{h}_{t+k+1}^k)$. The total expected benefit from a decision to invest today then is a sum of such price differences, weighted according to the probability of realizing $\mathbf{h}_{t+k+1}^k$ and accounting for an appropriate discounting.

So, we can compute the benefit of an investment for each history. Then, the reputational equilibrium exists if and only if the cost of investment is less than the minimum of benefits over all histories. We summarize this in the next lemma, which is a general statement of Lemma 1.

Lemma 1. *For an individual firm, there exists a constant $\hat{\mathbf{c}}^{ind} > 0$ such that the reputatoinal equilibrium exists if and only if $c \leq \hat{\mathbf{c}}^{ind}$ where*

$$\hat{\mathbf{c}}^{ind} = \min_{\mathbf{h}_t^{T-1}} \bar{c}^{ind}(\boldsymbol{h}_t^{T-1}) := \frac{\delta \Delta \pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f} \in \{G, \varnothing, B\}^k} Pr(\mathbf{f})(p(\mathbf{h}_t^{T-k-1} G \mathbf{f}) - p(\mathbf{h}_t^{T-k-1} B \mathbf{f})) \Big). \quad (7)$$

Proof. [Proof of Lemma 1] As in Lemma 1, we obtain an expression for the cutoff in terms of price differences. We find it useful to define a new value function before the consumer's

visit. Let $Z(\mathbf{h}_t)$ be the expected payoff to the firm in equilibrium:

$$Z^{\text{ind}}(\mathbf{h}_t) \equiv \tfrac{1}{2}(p(\mathbf{h}_t) - c) + \delta \left(\frac{\pi_H}{2} \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) + \frac{1-\pi_H}{2} \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}B) + \frac{1}{2} \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}\varnothing) \right).$$

As the consumer visits the firm with probability $\frac{1}{2}$, the firm's expected period-t profit is $\frac{1}{2}(p(\mathbf{h}_t) - c)$. The expected future payoff depends on the realized outcome in the current period. The firm produces outcomes G, B, $\varnothing$ with probabilities $\frac{\pi_H}{2}, \frac{1-\pi_H}{2}, \frac{1}{2}$, respectively.

Once the firm is visited, it should be optimal for the firm to invest always, i.e., $V(\mathbf{h}_t) \geq V(\mathbf{h}_t; \text{not})$ for all $\mathbf{h}_t \in \mathcal{H}^{\text{ind}}$ where

$$V^{\text{ind}}(\mathbf{h}_t) = p(\mathbf{h}_t) - c + \delta(\pi_H \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) + (1-\pi_H) \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}B)),$$
$$V^{\text{ind}}(\mathbf{h}_t; \text{not}) = p(\mathbf{h}_t) + \delta(\pi_L \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) + (1-\pi_L) \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}B)).$$

By investing in quality, the firm is able to produce a good outcome with a greater probability π_H, which improves the future payoffs. Then, the condition for the existence of the reputational equilibrium can be expressed as a cutoff-rule; the invest cost is always less than its benefit. So,

$$c \leq \hat{\mathbf{c}}^{\text{ind}} := \delta(\pi_H - \pi_L) \cdot \min_{\mathbf{h}_t^{T-1} \in \{G, \varnothing, B\}^{T-1}} \Delta Z^{\text{ind}}(\mathbf{h}_t^{T-1}), \tag{8}$$

where $\Delta Z^{\text{ind}}(\mathbf{h}_t^{T-1}) := Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) - Z^{\text{ind}}(\mathbf{h}_t^{T-1}B)$. The firm is able to receive a higher price in the next T periods due to the good outcome produced today. For this reason, $\Delta Z(\mathbf{h}_t^{T-1})$ is a present-discounted weighted-sum of price premiums, as we saw in the analysis for two-period memory:

The future payoff, conditional on producing a good outcome, is

$$
\begin{aligned}
Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) &= \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\{G,\varnothing,B\}^k} \Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-c)}_{\text{first } T \text{ periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j} \sum_{\mathbf{g}\in\{G,\varnothing,B\}^T} \Pr(\mathbf{g})(p(\mathbf{g})-c)}_{\text{after } T \text{ periods}}. \\
&= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k \left(\sum_{i+j+l=k} \left(\frac{\pi_H}{2}\right)^i \left(\frac{1-\pi_H}{2}\right)^j \left(\frac{1}{2}\right)^l \left(\sum_{N_G(\mathbf{f})=i, N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}G\mathbf{f}) \right) - c \right) \\
&+ \frac{1}{2}\delta^T \sum_{k=0}^{\infty}\delta^k \left(\sum_{i+j+l=T} \left(\frac{\pi_H}{2}\right)^i \left(\frac{1-\pi_H}{2}\right)^j \left(\frac{1}{2}\right)^l \left(\sum_{N_G(\mathbf{g})=i, N_B(\mathbf{g})=j} p(\mathbf{g}) \right) - c \right).
\end{aligned}
$$

Given a history $\mathbf{h}_t^{T-1}G$, the relevant history k periods later becomes $\mathbf{h}^{T-k-1}G\mathbf{f}$. That is, consumers replace oldest k memories with a new memory realized throughout k periods, i.e., $\mathbf{f} \in \mathcal{H}^k$. Conditional on the realization of $\mathbf{f}$, the firm's period-profit is $p(\mathbf{h}^{T-k-1}G\mathbf{f}) - c$. This realization occurs with a probability denoted by $\Pr(\mathbf{f})$. Accouting for these probabilities and discounting, we obtain the first double sum in the equation. Once T periods have passed and consumers no longer remember the good outcome of the investment made in period t, the firm's relevant history can be any $\mathbf{g} \in \mathcal{H}^T$. So, we obtain the second double sum by weighting and discounting each period-profit appropriately. The firm receives a period-profit if and only if the consumer visits, and therefore we divide the whole expression by 2.

To compute $\Pr(\mathbf{f})$, counting the number of good, bad and empty histories is just enough, as the order of each outcome does not matter. Let $N_h(\mathbf{h}_t)$ for $h \in \mathcal{H}$ and $\mathbf{h}_t \in \mathcal{H}^T$ be the count of an outcome of type h in the T-period history $\mathbf{h}_t$. For example, $N_G(G\varnothing G) = 2$, $N_B(G\varnothing G) = 0$ and $N_\varnothing(G\varnothing G) = 1$. Suppose $N_G(\mathbf{f}) = i$, $N_B(\mathbf{f}) = j$, and $N_\varnothing(\mathbf{f}) = l$, respectively, such that $i + j + l = k$. Then, $\Pr(\mathbf{f}) = (\frac{\pi_H}{2})^i \cdot (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l$. The next two lines in the equation are results of simply plugging in these probabilities.

Likewise, the future payoff to the firm if it produced a bad outcome would be

$$\begin{aligned} Z^{\text{ind}}(\mathbf{h}_t^{T-1}B) &= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{N_G(\mathbf{f})=i,N_B(\mathbf{f})=j}p(\mathbf{h}^{T-1-k}B\mathbf{f})\right)-c\right) \\ &\quad +\frac{1}{2}\delta^T\sum_{k=0}^{\infty}\delta^k\left(\sum_{i+j+l=T}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{N_G(\mathbf{g})=i,N_B(\mathbf{g})=j}p(\mathbf{g})\right)-c\right). \end{aligned}$$

Therefore, subtracting the two gives

$$\begin{aligned} \Delta Z^{\text{ind}}(\mathbf{h}_t^{T-1}) &= \tfrac{1}{2}\cdot\textstyle\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\{G,\varnothing,B\}^k}\Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-p(\mathbf{h}^{T-k-1}B\mathbf{f})) \\ &= \tfrac{1}{2}\cdot\textstyle\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\sum_{N_G(\mathbf{f})=i,N_B(\mathbf{f})=j}(p(\mathbf{h}^{T-1-k}G\mathbf{f})-p(\mathbf{h}^{T-1-k}B\mathbf{f}))\right) \end{aligned}$$

Plugging this into (8) completes the proof. □

To obtain an explicit expression for $\hat{c}^{\text{ind}}$, we need to uncover the minimum operator by identifying the binding history for different parameter regions. As in the two-period memory case, we focus on two special signal structures: exclusive knowledge ($\pi_L = 0$) and quality control ($\pi_H = 1$). The former provides an environment where building an extremely high level of reputation is easy for a competent firm, as one good outcome completely reveals its type. Therefore, we can attain the minimum by choosing a history that has a lasting damage to the firm's incentives. This implies that any history $\mathbf{h}_t^{T-1}$ with $h_{t-1} = G$ does the job. Since the most recent outcome in the history is good, consumers know perfectly the firm's type to be good until $t = T-1$. This eliminates all the benefits to be realized until period $t+T-1$. The only expression that survives in equation (7) is the very last period $(t+T)$ when $h_{t-1} = G$ will have been forgotten. As this benefit is discounted by δ^T, a longer history clearly hurts investment incentives for an individual brand.

Under the structure of quality control ($\pi_H = 1$), one bad outcome completely reveals a firm to be an incompetent type. Then, similarly, any history with $h_{t-1} = B$ attains the minimum because it puts a bad stamp on the brand for until period $t+T-1$. Then, all benefits other than ones to be realized in the very last period $(t+T)$, again discounted by

δ^T.

Therefore, $\lim_{\pi_L \to 0} \hat{\mathbf{c}}^{\text{ind}} = \lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(\mathbf{h}_t)$ where $h_{t-1} = G$, and $\lim_{\pi_H \to 1} \hat{\mathbf{c}}^{\text{ind}} = \lim_{\pi_H \to 1} \bar{c}^{\text{ind}}(\mathbf{g}_t)$ where $g_{t-1} = B$. We state next lemma with characterization of the cutoff once we take limits for μ, as we will use these cutoffs for comparison later.[18]

Lemma 2. *(i) In an the environment with exclusive knowledge ($\pi_L = 0$), a history in which the most recent outcome is G attains $\hat{\mathbf{c}}^{ind}$. If μ is close to 0,*

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\hat{\mathbf{c}}^{ind}}{1-\mu} = \frac{\delta^T \pi_H^2}{2(1-\pi_H)} \tag{9}$$

(ii). In an environment with quality control ($\pi_H = 1$), a history in which the most recent outcome is B attains $\hat{\mathbf{c}}^{ind}$. If μ is close to 0,

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\hat{\mathbf{c}}^{ind}}{\mu} = \frac{\delta^T (1-\pi_L)^2}{2^T \pi_L} \cdot \left(\frac{1+\pi_L}{\pi_L}\right)^{T-1}. \tag{10}$$

Proof. [Proof of Lemma 2] (Identify the binding constraint for two cases and then compute the cutoff-level.) As the exact cutoff level involves a minimum operator, we need to compare $\Delta Z(\mathbf{h}_t^{T-1})$ for all $\mathbf{h}_t^{T-1} \in \{G, B, \varnothing\}$. Obtaining an explicit formula for it is not feasible. Instead, we focus on two special signal structures–$\pi_L = 0$, $\pi_H \in (0,1)$ and $\pi_H = 1$, $\pi_L \in (0,1)$.

First, suppose $\pi_L = 0$, $\pi_H \in (0,1)$. This is the case of exclusive technology where a good outcome reveals the firm to be competent. So, $\mu(\mathbf{h}) = 1$ if and only if $N_G(\mathbf{h}) \geq 1$. Here, the price $p(\mathbf{h}) = \pi_H \cdot \mu(\mathbf{h})$. So,

$$\begin{aligned} p(\mathbf{h}^{T-1-k} G\mathbf{f}) - p(\mathbf{h}^{T-1-k} B\mathbf{f}) &= \pi_H \cdot (\mu(\mathbf{h}^{T-1-k} G\mathbf{f}) - \mu(\mathbf{h}^{T-1-k} B\mathbf{f})) \\ &= \pi_H \cdot (1 - \mu(\mathbf{h}^{T-1-k} B\mathbf{f})). \end{aligned}$$

This vanishes if and only if $N_G(\mathbf{h}^{T-1-k} B\mathbf{f}) \geq 1$, i.e. there is at least one good outcome

[18] Please see the appendix for the cutoffs prior to taking limits of μ.

in this history. To find a history that minimizes $\Delta Z(\cdot)$, we want as many of the price difference as possible to vanish. For this purpose, it suffices to have $h_{-1} = G$. Recall h_{-1} is the outcome produced a period before the focal investment decision. So, the good outcome reveals the firm's competence until it is forgotten T periods later. So, with $h_{-1} = G$, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) = 0$ for all $\mathbf{f} \in \mathcal{H}^k$ for $0 \leq k \leq T-2$. For $k = T-1$, h_{-1} is forgotten and the relevant price premium is $p(G\mathbf{f}) - p(B\mathbf{f})$. So, for $h_{-1} = G$,

$$\Delta Z^{\text{ind}}(\mathbf{h}) \rightarrow_{\pi_L \rightarrow 0} \frac{1}{2} \cdot \delta^{T-1} \sum_{\mathbf{f} \in \mathcal{H}^{T-1}} \Pr(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f}))$$

That is, all benefits other than the one realized in the last period vanish. And, this part is independent of $\mathbf{h}$, the history at the time of investment decision. Therefore, $h_{-1} = G$ indeed attains the minimum for $\Delta Z(\cdot)$.

Clearly, $p(G\mathbf{f}) - p(B\mathbf{f})$ again vanishes for any $N_G(\mathbf{f}) \geq 1$. Therefore, terms that survive in the equation above are $\mathbf{f}$ of length $T-1$ that only consist of B and/or $\varnothing$. Therefore,

$$\begin{aligned}
\Delta Z^{\text{ind}}(\mathbf{h}) \quad \rightarrow_{\pi_L \rightarrow 0} \quad & \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \pi_H \left(\hat{\mu}(GB^j \varnothing^{T-1-j}) - \hat{\mu}(B^{j+1} \varnothing^{T-1-j}) \right) \right) \\
= \quad & \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \pi_H \left(1 - \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1} + 1 - \mu} \right) \right) \\
= \quad & \frac{\pi_H(1-\mu)}{2^T} \cdot \delta^{T-1} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)} \right).
\end{aligned}$$

The first equality holds because $\hat{\mu}(GB^j \varnothing^{T-1-j}) = 1$ because a good history causes a full revelation, and $\hat{\mu}(B^{j+1} \varnothing^{T-1-j}) = \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1}+1-\mu}$. Simply plugging into (8) proves the lemma for $\pi_L = 0$ and $\pi_H \in (0,1)$. In particular, $\lim_{\mu \rightarrow 1} \lim_{\pi_L \rightarrow 0} \frac{\Delta Z^{\text{ind}}(\mathbf{h})}{1-\mu} = \frac{\pi_H}{2(1-\pi_H)} \cdot \delta^{T-1}$.

Now, consider the case where $\pi_H = 1$ and $\pi_L \in (0,1)$. Here, a bad outcome is revealing of a firm's incompetence. Therefore, $\mu(\mathbf{h}) = 0$ if and only if $N_B(\mathbf{h}) \geq 1$, and $p(\mathbf{h}) = \pi_L$. We omit details for this case, as it is very similar to the previous case.

From (B.1), $h_{-1} = B$ attains the minimum for $\Delta Z^{\text{ind}}(\cdot)$. Then, all price premiums other

than the ones to be realized in the last period vanish. Therefore,

$$\begin{aligned}
\Delta Z^{\text{ind}}(\mathbf{h}) \quad &\to_{\pi_H \to 1} \quad \frac{1}{2} \cdot \delta^{T-1} \sum_{\mathbf{f} \in \mathcal{H}^{T-1}} \Pr(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f})) \\
&= \quad \frac{\delta^{T-1}(1-\pi_L)}{2} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1}{2})^j (\frac{1}{2})^{T-1-j} \left(\hat{\mu}(G^{j+1} \varnothing^{T-1-j}) - \hat{\mu}(BG^j \varnothing^{T-1-j}) \right) \right) \\
&= \quad \frac{\delta^{T-1}(1-\pi_L)\mu}{2^T} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{1}{\mu + (1-\mu)\pi_L^{j+1}} \right).
\end{aligned}$$

Plugging this into (8) completes the proof. In particular, $\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\Delta Z^{\text{ind}}(\mathbf{h})}{\mu} = \frac{\delta^{T-1}(1-\pi_L)}{2^T \pi_L} \cdot (\frac{1+\pi_L}{\pi_L})^{T-1}$. □

As we see in equations (9) and (10), the expected benefit to be realized in the last period is a weighted sum, depending on realization of $\mathbf{f}$, the future outcomes following the focal investment decision at period t. The price differences are of the form $p^{\text{ind}}(G\mathbf{f}) - p^{\text{ind}}(B\mathbf{f})$, where $\mathbf{f} \in \{G, \varnothing, B\}^{T-1}$. Under $\pi_L = 0$, if any outcome in $\mathbf{f}$ is G, the difference vanishes, as one good outcome reveals the firm to be competent. So, the summation accounts for the cases where $\mathbf{f} \in \{\varnothing, B\}^{T-1}$, i.e. only bad or empty outcomes constitute $\mathbf{f}$. Likewise, under $\pi_H = 1$, the price difference vanishes if and only if there is a B in $\mathbf{f}$. So, (10) sums over the cases $\mathbf{f} \in \{G, \varnothing\}^{T-1}$.[19]

B.2 Collective brand

A longer memory may have a similar adverse effect on short-run incentives for collective brands. However, as we saw in the analysis of the two-period model, consumers' limited observability for a collective brand alleviates this problem; as consumers cannot observe history at firm-level, they can never learn perfectly about the types of two firms in the group. Therefore, a competent firm can always improve the brand reputation by investing in quality.

[19] See more details in the appendix.

The next lemma establishes the necessary and sufficient condition for the existence of reputational equilibrium. Let $\Pr(\mathbf{f};\theta)$ for $\mathbf{f} \in \{G,B\}^k$ and $\theta \in \{C,I\}$ with $0 \leq k \leq T$ be the probability that the brand of type θ produces a sequence of outcome $\mathbf{f}$ in k periods if a competent firm always invests.

Lemma 3. *For a competent firm within a collective brand, there exists a constant $\hat{c} > 0$ such that a* **RE** *exists if and only if $c \leq \hat{c}$ where*

$$\hat{\mathbf{c}}^{col} = \min_{\boldsymbol{h}_t^{T-1},\theta} \bar{c}^{col}(\boldsymbol{h}_t^{T-1},\theta) := \frac{\delta\Delta\pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f}\in\{G,B\}^k} Pr(\mathbf{f};\theta)(p(\mathbf{h}_t^{T-k-1}G\mathbf{f}) - p(\mathbf{h}_t^{T-k-1}B\mathbf{f}))\Big), \tag{11}$$

where $\mathbf{h}_t^{T-1} \in \{G,B\}^{T-1}$ and $\theta \in \{C,I\}$.

Proof. [Proof of Lemma 3] As this lemma is a straightforward generalization of lemma 3, we omit many details. Also, we adopt notation from the proof for 1. Let $Z_\theta^{\text{col}}(\mathbf{h}_t)$ denote the payoff to a competent firm of a collective brand before the customer's visit. $\theta \in \{C,I\}$ denotes the other firm's type, which determines the brand's type, $s \in \{CC, CI\}$.

$$Z_\theta^{\text{col}}(\mathbf{h}_t) \equiv \underbrace{\frac{1}{2}(p(\mathbf{h}_t) - c)}_{\text{expected current period}} + \delta \left(\underbrace{\frac{\pi_H + \pi(\theta)}{2} \cdot Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1 - \frac{\pi_H + \pi(\theta)}{2}) \cdot Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)}_{\text{future profit}} \right).$$

In the current period the firm makes $p(\mathbf{h}_t) - c$ if visited and 0 otherwise. In the next period, the brand will face a history $\mathbf{h}_t^{T-1}G$ or $\mathbf{h}_t^{T-1}B$ depending on today's investment outcome, which also depends on the type of the other firm. So, on average, the firm produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B otherwise.

Once the firm is visited, it should be optimal for the firm to invest always. After a history $\mathbf{h}_t$, a firm's payoff conditional on being visited is denoted by $V_\theta^{\text{col}}(\mathbf{h}_t)$. Then, we need

$V_\theta^{\text{col}}(\mathbf{h}_t) \geq V_\theta^{\text{col}}(\mathbf{h}_t; \text{not})$ for all $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$.

$$V_\theta^{\text{col}}(\mathbf{h}_t) = p(\mathbf{h}_t) - c + \delta(\pi_H \cdot Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1-\pi_H) \cdot Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)),$$
$$V_\theta^{\text{col}}(\mathbf{h}_t; \text{not}) = p(\mathbf{h}_t) + \delta(\pi_L \cdot Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1-\pi_L) \cdot Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)).$$

This is equivalent to

$$c \leq \bar{\mathbf{c}}^{\text{col}} := \delta(\pi_H - \pi_L) \cdot \min_{\mathbf{h}_t^{T-1} \in \{G,B\}^{T-1}} \Delta Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}), \tag{12}$$

where $\Delta Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}) := Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) - Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)$.

The future payoff, conditional on producing an outcome of either G or B, is

$$Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k} \Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j} \sum_{\mathbf{g}\in\mathcal{H}^T} \Pr(\mathbf{g};\theta)(p(\mathbf{g}) - c)}_{\text{After } T \text{ Periods}},$$
$$Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k} \Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}B\mathbf{f}) - c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j} \sum_{\mathbf{g}\in\mathcal{H}^T} \Pr(\mathbf{g};\theta)(p(\mathbf{g}) - c)}_{\text{After } T \text{ Periods}}$$

In each period, the brand produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B with the complementary probility. Therefore, for any $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$, if $N_G(\mathbf{h}_t) = i$ and $N_B(\mathbf{h}_t) = j = t - i$, $\Pr(\mathbf{f};\theta) = (\frac{\pi_H+\pi(\theta)}{2})^i(1 - \frac{\pi_H+\pi(\theta)}{2})^j$.

Therefore, subtracting the two gives

$$\begin{aligned}\Delta Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}) &= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k} \Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - p(\mathbf{h}^{T-k-1}B\mathbf{f})) \qquad (13)\\ &= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k \left(\sum_{i+j=k}(\frac{\pi_H+\pi(\theta)}{2})^i(1-\frac{\pi_H+\pi(\theta)}{2})^j \left(\sum_{N_G(\mathbf{f})=i}(p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}))\right)\right)\end{aligned}$$

Plugging this into (12) completes the proof. □

This lemma generalizes lemma 3. The cutoff now depends on the type of the other firm,

as it affects realization of future outcomes $\mathbf{f}$ through $\Pr(\mathbf{f};\theta)$. Also, prices here are different from those in the individual brand because conditional on a history, posterior beliefs are different.

None of these price differences in equation (11) vanish even for $\pi_L = 0$ and $\pi_H = 1$. And, which history and type provide the binding constraint is less clear for a collective brand. To characterize the cutoff level, we take limits for μ, the prior belief.

First, for $\pi_L = 1$, consider μ close to 1. Then, a good outcome is informative. However, the informativeness of each additional good outcome must be decreasing. For example, having one good outcome compared to none is quite desirable, as it reveals the existence of at least one competent firm. But, having a fifth good outcome in the history in addition to an existing four is not as appealing, as consumers already believe with a high probability that both firms are competent. So, in this parameter region, the binding constraint would be provided by an environment that produces as many good outcomes as possible. Naturally, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would do the job.

Second, for $\pi_H = 1$, let focus on μ close to 0. Then, while a bad outcome is informative, it's informativeness decreases as there are more bad outcomes in the history. So, the binding condition would be provided by $\mathbf{h}_t^{T-1} = B^{T-1}$ and $\theta = I$, as together they produce as many bad outcomes as possible in the brand's history.

Then, we can compute the cutoff levels explicitly:

Lemma 4. *(i) Under the environment of exclusive technology ($\pi_L = 0$), if μ is close to 1, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(G^{T-1}, C)$ and*

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\hat{\mathbf{c}}^{col}}{1-\mu} = \frac{\delta\pi_H^2}{2^{T+1}(1-\pi_H)}\cdot\frac{1-(2\delta)^T}{1-2\delta} \tag{14}$$

(ii) Under the quality control ($\pi_H = 1$), if μ is close to 0, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(B^{T-1}, I)$ and

$$\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\hat{\mathbf{c}}^{col}}{\mu} = \frac{\delta(1-\pi_L)^2}{2^{T+1}\pi_L}\cdot\frac{1-(\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L}} \tag{15}$$

Proof. [Proof of Lemma 4] The exact cutoff levels in lemma 3 is a discounted sum of price premiums over T periods. It is not feasible to obtain an explicit expression for general parameter regions. So, we again focus on two parameter regions: $\pi_L = 0$ and $\pi_H = 1$. Before we shift our focus to these cases, we find it useful to understand posterior beliefs denoted by $\eta(\cdot)$. Facing a collective brand, consumers update beliefs over types of the brand, $s \in \{CC, CI, IC, II\}$, and use this to compute the probability of visiting a competent firm: $\eta(\cdot) = \eta_{CC}(\cdot) + \frac{1}{2}(\eta_{CI}(\cdot) + \eta_{IC}(\cdot))$. So, $\eta(\mathbf{h}_t)$, if $N_G(\mathbf{h}_t) = i$, is

$$\eta(\mathbf{h}_t) = \frac{\mu^2 \cdot \pi_H^i (1-\pi_H)^{T-i} + \mu(1-\mu) \cdot (\frac{\pi_H+\pi_L}{2})^i (1-\frac{\pi_H+\pi_L}{2})^{T-i}}{\mu^2 \cdot \pi_H^i (1-\pi_H)^{T-i} + 2\mu(1-\mu) \cdot (\frac{\pi_H+\pi_L}{2})^i (1-\frac{\pi_H+\pi_L}{2})^{T-i} + (1-\mu)^2 \cdot \pi_L^i (1-\pi_L)^{T-i}} \tag{16}$$

It is infeasible to obtain an explicit expression for $\Delta Z_\theta(\cdot)$, not to mention the overall cutoff, $\bar{c}^{\text{col}}$. As we did in previous analyses, we i) focus on two signal structures ($\pi_L = 0$ and $\pi_H = 1$), ii) identify the binding history and the brand type, and iii) obtain a lower bound $\underline{c}^{\text{col}}$ for the cutoff.

First, consider the case $\pi_L = 0$. Then, after a history $\mathbf{h}_t$, the consumer pays $p(\mathbf{h}_t) = \eta(\mathbf{h}_t) \cdot \pi_H$. The reputational benefit realized in each period is the price difference made available by one more good outcome in the history, and thus is of a form $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, where $N_G(\mathbf{h}^{T-1-k}G\mathbf{f}) = N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) + 1$. And, here we claim that this difference is decreasing in i for a large enough μ. That is, when $\pi_L = 0$ and μ is large, the price premium reduces as the number of good outcomes becomes large. If this were true, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would provide the minimum for $\Delta Z_\theta(\mathbf{h}_t^{T-1})$, as these two conditions both places the brand under histories with more good outcomes. We formally state this and prove:

Claim 1. Suppose $\pi_L = 0$ and μ is close to 1. And let $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) = i$. Then, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$ is decreasing in i. Then, the price premium from the investment is low when there are many good outcomes in the history. So, $\boldsymbol{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ attains the minimum for $\Delta Z_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\bar{c}^{col}$.

The intuition is the following. As long as there is a good outcome in the history, con-

sumers believe the brand has either one or two competent firms. But, as they see more good outcomes, they become more convinced that both firms are competent. As more good outcomes resolve consumers' uncertainty, the price difference becomes small. Mathematically,

$$\begin{aligned}
\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2) &= \frac{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}} \\
&\quad - \frac{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&= \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&\quad - \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}
\end{aligned}$$

Then, taking $\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{1-\mu}$ to a limit as $\mu \to 1$,

$$\begin{aligned}
\lim_{\mu\to 1} \frac{\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)}{1-\mu} &= \frac{\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\pi_H^{i}(1-\pi_H)^{T-i}} - \frac{\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\pi_H^{i+1}(1-\pi_H)^{T-i-1}} \\
&= \frac{1}{(1-\pi_H)2^{i+1}}\left(\frac{1-\frac{\pi_H}{2}}{1-\pi_H}\right)^{T-i-1},
\end{aligned}$$

which is clearly decreasing in i. Therefore, for any positive integer T, there is a $\bar{\mu}$ close enough to 1 so that the difference in beliefs (and thus prices) is decreasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = G^{T-1}$ and $\theta = C$ to compute:

$$\begin{aligned}
\lim_{\mu\to 1} \frac{\Delta Z_C^{\text{col}}(G^{T-1})}{1-\mu} &= \frac{\pi_H}{2} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j=k} \pi_H^i (1-\pi_H)^j \left(\sum_{N_G(\mathbf{f})=i} (\eta(\mathbf{h}^{T-1-k} G\mathbf{f}) - \eta(\mathbf{h}^{T-1-k} B\mathbf{f})) \right) \right) \\
&= \frac{\pi_H}{2(1-\pi_H)} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j=k} \pi_H^i (1-\pi_H)^j \binom{k}{i} \frac{1}{2^{T-k+i}} \left(\frac{1-\frac{\pi_H}{2}}{1-\pi_H}\right)^j \right) \\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \sum_{k=0}^{T-1} (2\delta)^k \\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta}
\end{aligned}$$

Next, we consider the case $\pi_H = 1$. Then, the price consumer pays after a history $\mathbf{h}_t$ is $p(\mathbf{h}_t) = \eta(\mathbf{h}_t) + (1 - \eta(\mathbf{h}_t))\pi_L$. In this setting, a bad outcome is very informative, as it reveals existence of an incompetent firm in the brand. And, intuitively as there are more bad outcomes in the history, informativeness of each bad outcome decrease. Therefore, the price premium to be realized k period after the focal investment decision conditional on the new outcomes $\mathbf{f}$ is $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, and this decreases in i, where $i = N_G(\mathbf{h}^{T-1-k}B\mathbf{f})$. We state it formally in the next claim.

Claim 2. Suppose $\pi_H = 0$ and μ is close to 0. And let $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) = i$. Then, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$ is increasing in i. Then, $\boldsymbol{h}_t^{T-1} = B^{T-1}$ and $\theta = I$ attains the minimum for $\Delta Z_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\bar{c}^{col}$.

$$\begin{aligned}
\eta(G^T) &= \frac{\mu^2 + \mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T}{\mu^2 + 2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T + (1-\mu)^2\cdot\pi_L^T} \\
\eta(\mathbf{h}) &= \frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}}{2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, $\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2) =$

$$\begin{aligned}
&\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1} + (1-\mu)^2\cdot\pi_L^{i+1}(1-\pi_L)^{T-i-1}} \\
&- \frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i}(\frac{1-\pi_L}{2})^{T-i}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i}(\frac{1-\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^{i}(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, taking $\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{\mu}$ to a limit as $\mu \to 0$,

$$\begin{aligned}
\lim_{\mu\to 1}\frac{\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)}{\mu} &= (\frac{1+\pi_L}{2\pi_L})^{i+1}\frac{1}{2^{T-i-1}} - (\frac{1+\pi_L}{2\pi_L})^{i}\frac{1}{2^{T-i}} \\
&= \frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}.
\end{aligned}$$

This is clearly increasing in i. Therefore, there is a $\bar{\mu}_{\pi_H=1}$ close enough to 0 so that the difference in beliefs (and thus prices) is increasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = G^{T-1}$ and $\theta = C$ to compute:

$$\begin{aligned}
\lim_{\mu\to 0}\frac{\Delta Z}{\mu} &= \frac{1-\pi_L}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\left(\sum_{N_G(\mathbf{f})=i}(\frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}})\right)\right)\\
&= \frac{1-\pi_L}{2^{T+1}}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\binom{k}{i}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}\right)\\
&= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\sum_{k=0}^{T-1}\frac{\delta^k}{2^k}\left(\frac{1+3\pi_L}{\pi_L}\right)^k\\
&= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})}
\end{aligned}$$

□

Even in the limits, benefits of investment for a collective brand do not vanish, and the cutoff turns out to be a sum of what turns out to be a finite geometric sequence. Unlike the cutoff for an individual brand, the cutoff is not discounted by δ^T, so it decreases in T as a much slower rate. This highlights the advantage of collective brands over individual ones.

B.3 Comparing Individual and Collective Brands

It remains to find out when $\hat{\mathbf{c}}^{\text{col}}$ is greater than $\hat{\mathbf{c}}^{\text{ind}}$ for two special parameter regions by comparing equations (9) and (14), and (10) and (15).

Proposition 10. *(i) If $\pi_L = 0$, and μ close to 1, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$ if either*

$$\delta < \frac{1}{2},\ \text{or } \delta > \frac{1}{2} \text{ and } (2\delta)^T > \frac{\delta}{1-\delta}. \tag{17}$$

(ii) If $\pi_H = 1$, and μ close to 0, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$ if

$$\frac{1}{2}\cdot\frac{1-\left(\frac{\delta(1+3\pi_L)}{2\pi_L}\right)^T}{1-\frac{\delta(1+3\pi_L)}{2\pi_L}} > \left(\frac{\delta(1+\pi_L)}{\pi_L}\right)^{T-1} \tag{18}$$

Proposition 8 shows that for any history length, we can find regions where a collective brand sustains the reputational equilibrium better than an individual brand does. In fact, a longer memory expands the region of δ that supports our result for the case of $\pi_L = 0$. This is because when $\delta > \frac{1}{2}$, $2\delta > 1$, so the left-hand side increases in T. For example, if $T = 2$, $\hat{\mathbf{c}}^{\text{col}} > \hat{\mathbf{c}}^{\text{ind}}$ for $\delta \in (0, \frac{1}{2})$, if $T = 3$, $\delta \in (0, \frac{1+\sqrt{5}}{4} \approx 0.809)$, and if $T = 4$, $\delta \in (0, 0.919)$.

The case of $\pi_H = 1$ is more complicated. Because the left-hand side is always increasing in T, (18) is more likely to hold if $\frac{\delta(1+\pi_L)}{\pi_L} \leq 1$. Otherwise, if $\frac{\delta(1+\pi_L)}{\pi_L} > 1$, the right-hand side diverges as T goes to infinity. So, in order for the condition to hold, the left-hand side must diverge at a faster rate. The left-hand side converges if and only if $\delta < \frac{2\pi_L}{1+3\pi_L}$. So, if $\frac{\pi_L}{1+\pi_L} < \delta < \frac{2\pi_L}{1+3\pi_L}$, the condition holds only for a small enough T. If $\delta > \frac{2\pi_L}{1+3\pi_L}$, we can show that the condition cannot hold for T too large.

Corollary 2. *If $\pi_L = 0$ and μ close to 1, for any $\delta \in (0, 1)$, there is a large enough $\bar{T}$ such that for all $T > \bar{T}$, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$. If $\pi_H = 1$ and μ close to 0, for any $\delta \in (0, \frac{\pi_L}{1+\pi_L}]$, there is a large enough $\tilde{T}$ such that for all $T > \tilde{T}$, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$. Otherwise, if $\delta \in (\frac{\pi_L}{1+\pi_L}, 1)$, there is a large enough $\hat{T}$ such that for all $T > \hat{T}$, $\hat{\mathbf{c}}^{ind} > \hat{\mathbf{c}}^{col}$.*